Cultivating Complex Analysis: Line integrals (3.1 part 1)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

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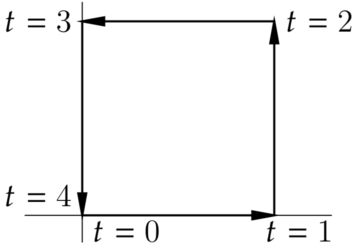
E.g., we say γ is in U or $\gamma \subset U$ if $\gamma\bigl([a,b]\bigr) \subset U$.

Example: Consider $\gamma\colon [0,4] \to \mathbb{C}$,

$$\gamma(t) = \begin{cases} t & \text{if } t \in [0,1], \\ 1 + i(t-1) & \text{if } t \in (1,2], \\ 3 - t + i & \text{if } t \in (2,3], \\ i(4-t) & \text{if } t \in (3,4]. \end{cases}$$



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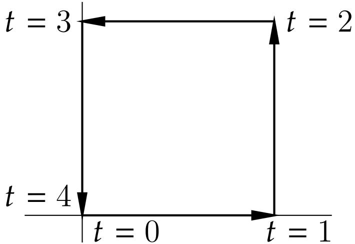

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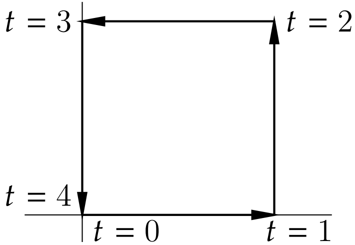

Note, for example, on $t \in (0,1)$, $\gamma'(t) = 1$, and so $\lim_{t\uparrow 1} \gamma'(t) = 1$.

Similarly $\lim_{t\downarrow 1} \gamma'(t) = i$, etc.

Given a piecewise-C^1 path $\gamma\colon [a,b] \to \mathbb{C}$ and a continuous function f on γ, we define the *line integral* (or *path integral*, *curve integral*, *contour integral*)

$$\int_\gamma f(z)\,dz \stackrel{\text{def}}{=} \int_a^b f\big(\gamma(t)\big)\gamma'(t)\,dt.$$

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The definition makes sense even if $\gamma'(t)$ is zero somewhere.

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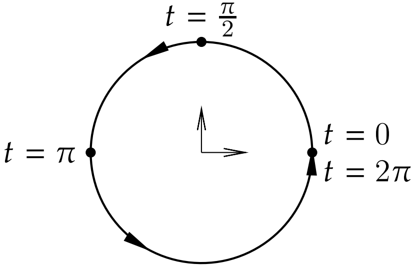

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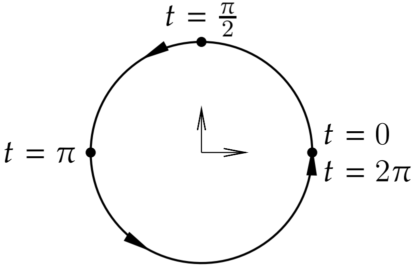
$$\int_\gamma z^n\, dz = \begin{cases} 2\pi i & \text{if } n = -1, \\ 0 & \text{otherwise.} \end{cases}$$



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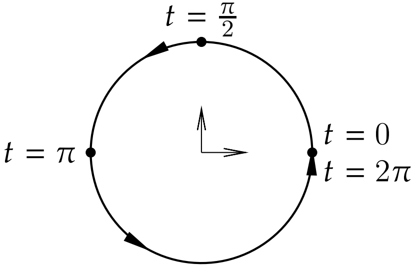

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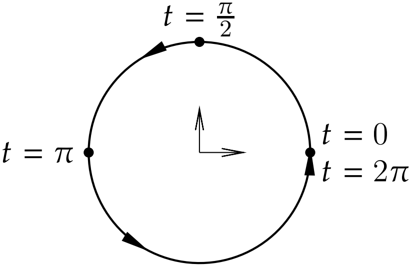

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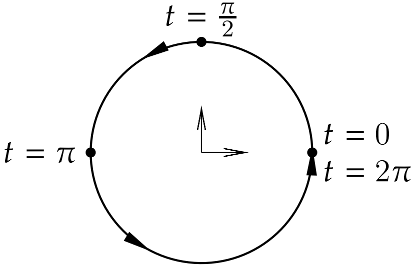

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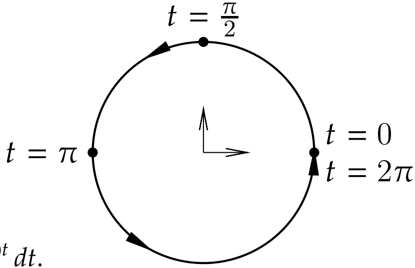

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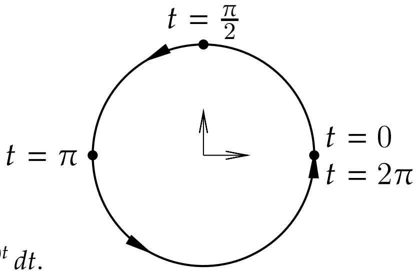

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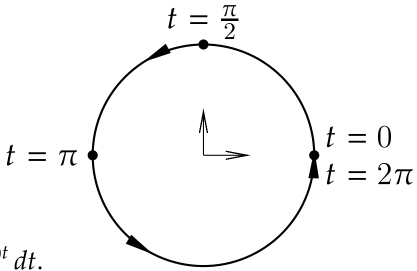

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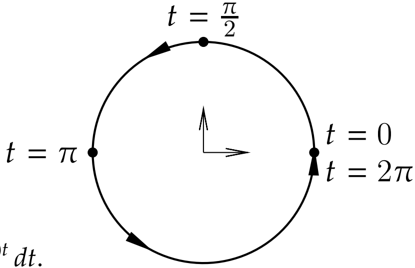

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Note that the value of the integral does not depend on r.

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In fact, if you also write $d\bar{z} = dx - i\,dy$, you can write any integral

$$\int_\gamma P\,dx + Q\,dy \qquad \text{as} \qquad \int_\gamma F\,dz + G\,d\bar{z}$$

and vice versa (exercise).