Cultivating Complex Analysis: Singularities and the Laurent series (5.2.2)

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The Laurent series at $z = 0$ is just ${}^{1}/_{z}$,
and all coefficients of order less than -1 are zero.

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Again, pole of order 1 at $z = 0$,
and all coefficients in the series of order less than -1 are zero.

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A pole of order 3,
and all coefficients in the series of order less than -3 are zero.

$$e^{1/z} = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n$$

has an essential singularity at $z = 0$,
and has nonzero coefficients of all negative orders.

It is not difficult to prove the general statement:

Proposition

Suppose $f : \Delta_r(p) \setminus \{p\} \to \mathbb{C}$ *is holomorphic, and*

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Hint: Laurent series is unique, and for a removable singularity equals the power series.

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Observation: If $P(z)$ is the principal part of $f(z)$ at p, then $f(z) - P(z)$ has a removable singularity at p.

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e^z has an essential singularity at infinity,
because $e^{1/z}$ has an essential singularity at 0.

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Hint: First do it for a simple pole.

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Exercise: Show that if $f : \mathbb{C} \to \mathbb{C}$ is an automorphism, then $f(z) = az + b$ for some constants $a \neq 0$ and b. Hint: Show that f has a simple pole at infinity.