Cultivating Complex Analysis: Convergence of sequences of holomorphic functions (3.4.2)

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A sequence of functions $f_n\colon U \to \mathbb{C}$ converges *uniformly on compact subsets* to $f\colon U \to \mathbb{C}$ if $f_n|_K$ converges uniformly to $f|_K$ for every compact $K \subset U$.

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$|x|^{1+1/n}$ is C^1 on $\mathbb{R}$ and converges uniformly on compact subsets to $|x|$.

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Suppose $U \subset \mathbb{C}$ *is open and* $f_n : U \longrightarrow \mathbb{C}$ *is a sequence of holomorphic functions converging uniformly on compact subsets to* $f : U \longrightarrow \mathbb{C}$.

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We can take the limit $n \to \infty$ underneath the integral.

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We still need to prove the "Moreover" bit:
That $\{f_n^{(\ell)}\}$ also converges uniformly on compact subsets.

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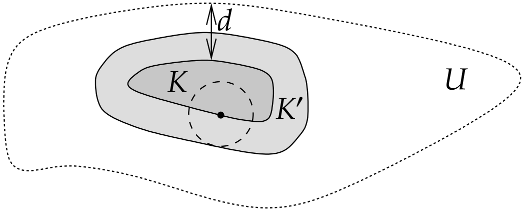

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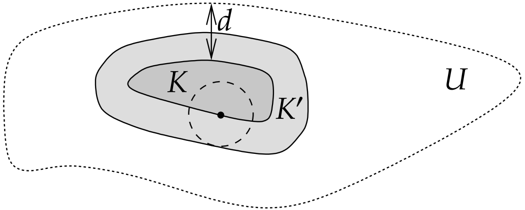

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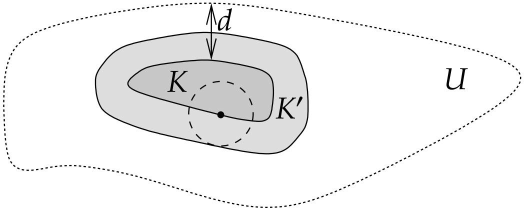

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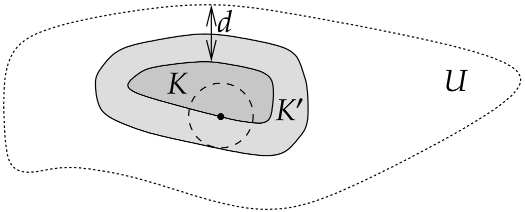

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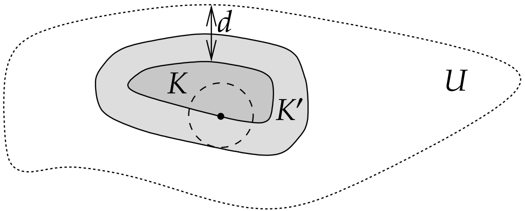

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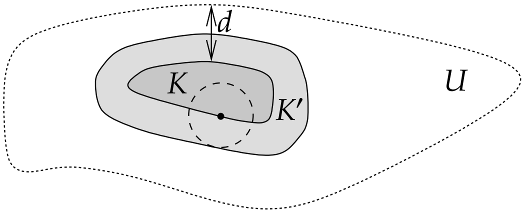

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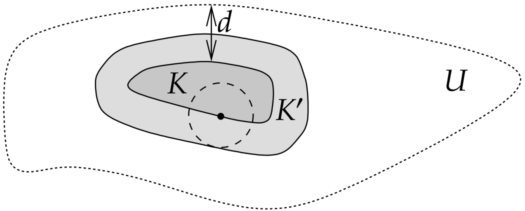

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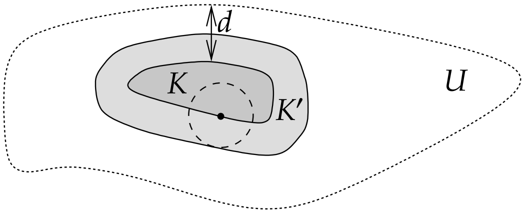

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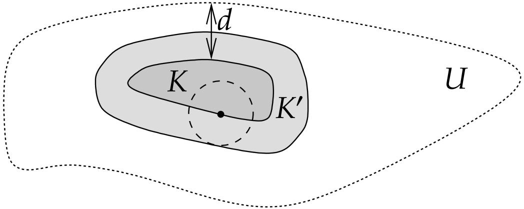

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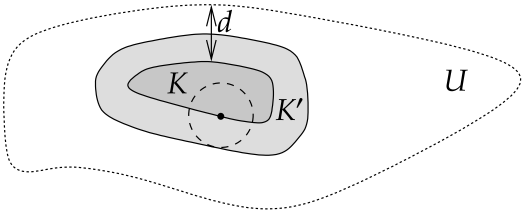

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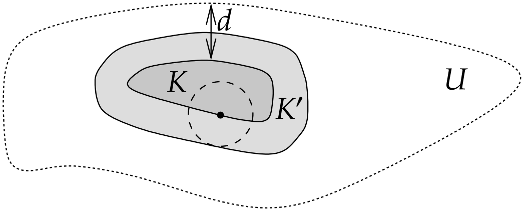

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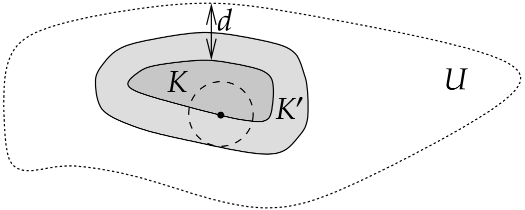

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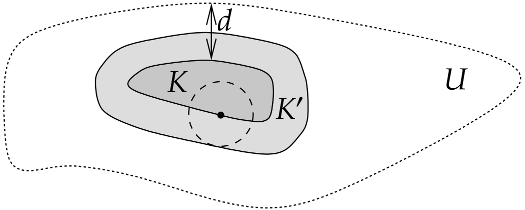

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For any $p \in K$, by Cauchy estimates on $f_n - f$ in $\Delta_{d/2}(p)$ (note $\partial\Delta_{d/2}(p) \subset K'$):

$$\left| f_n^{(\ell)}(p) - f^{(\ell)}(p) \right| \leq \frac{\ell! \, \|f_n - f\|_{\partial\Delta_{d/2}(p)}}{(d/2)^\ell} \leq \frac{\ell! 2^\ell}{d^\ell} \epsilon.$$

$\Rightarrow \{f_n^{(\ell)}\}$ converges uniformly to $f^{(\ell)}$ on K. $\square$

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Integration is a far nicer operation than differentiation, and for holomorphic functions, we can differentiate by integrating.