Cultivating Complex Analysis:
The exponential (as power series) (2.4.3)

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is the unique convergent power series at the origin such that $f(0) = 1$ *and* $f' = f$.

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Hence, $f(z) = C\exp(z)$ for some constant C. As $f(0) = \exp(0) = 1$, conclude $C = 1$. $\square$