Cultivating Complex Analysis: Holomorphic functions via integrals (3.4.1)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

Lemma

Suppose $U \subset \mathbb{C}$ *is open, and* $\psi \colon U \times [0,1] \to \mathbb{C}$ *is a continuous function such that for each fixed* $t \in [0,1]$*, the function* $z \mapsto \psi(z,t)$ *is holomorphic.*

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$h(z)$ is holomorphic by Morera. □

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OK, now we are done. $\square$

Corollary

Suppose $U \subset \mathbb{C}$ is open, Γ is a chain, and $\psi\colon U \times \Gamma \to \mathbb{C}$ is a continuous function such that for each fixed $w \in \Gamma$, the function $z \mapsto \psi(z, w)$ is holomorphic. Then

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$$Cf(z) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{\partial\Delta_r(p)} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

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For a random continuous f, Cf may not tend to f as we approach $\partial\Delta_r(p)$.

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then $f|_{\Delta_r(p)}$ is holomorphic.