Cultivating Complex Analysis: Cauchy–Riemann equations (2.1.2)

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Then $Df|_{z_0}$ corresponds to multiplication by $\xi = \frac{\partial u}{\partial x}\Big|_{z_0} + i\frac{\partial v}{\partial x}\Big|_{z_0} = \frac{\partial v}{\partial y}\Big|_{z_0} - i\frac{\partial u}{\partial y}\Big|_{z_0}$.

Consequently,

$$0 = \lim_{h \to 0} \frac{|f(z_0 + h) - f(z_0) - \xi h|}{|h|} = \lim_{h \to 0} \left| \frac{f(z_0 + h) - f(z_0)}{h} - \xi \right|,$$

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We proved that: If f is (real) differentiable at z_0, with $\frac{\partial u}{\partial x}\big|_{z_0} = \frac{\partial v}{\partial y}\big|_{z_0}$ and $\frac{\partial v}{\partial x}\big|_{z_0} = -\frac{\partial u}{\partial y}\big|_{z_0}$, then f is complex differentiable at z_0.

Conversely, if f is complex differentiable at z_0, then it is real differentiable at z_0 (the complex derivative $f'(z_0)$ gives the $Df|_{z_0}$).

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Proposition

Let $U \subset \mathbb{C}$ be open and $f = u + iv \colon U \to \mathbb{C}$ be a function. Then f is complex differentiable at $z_0 \in U$ if and only if f (real) differentiable at $z_0 \in U$ with $\frac{\partial u}{\partial x}\big|_{z_0} = \frac{\partial v}{\partial y}\big|_{z_0}$ and $\frac{\partial v}{\partial x}\big|_{z_0} = -\frac{\partial u}{\partial y}\big|_{z_0}$.

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Corollary

Let $U \subset \mathbb{C}$ be open and let $f = u + iv \colon U \to \mathbb{C}$ be a function such that $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial v}{\partial y}$ exist and are continuous (that is, f is continuously differentiable).

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$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \tag{1}$$

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$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \qquad \text{exists for all } z \in U.$$

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Remark: If only the partial derivatives exist but aren't continuous, the function may fail to be differentiable (or even continuous) and may not be holomorphic even if it satisfies (1).

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Hint: Note that $e^{x+iy} = e^x \cos y + ie^x \sin y$ and use the Corollary from previous slide.