Cultivating Complex Analysis: Linear fractional transformations (1.4 part 1)

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So takes the Riemann sphere to Riemann sphere.

It is an easy exercise that $f \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$ is bijective and continuous.

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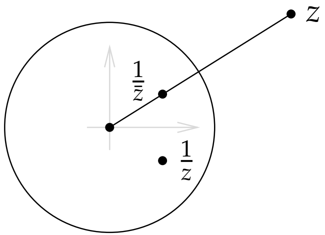
$$\frac{1}{|z|}e^{i\arg z} = \frac{|z|}{|z|^2}e^{i\arg z} = \frac{1}{\bar{z}}.$$

To get complex inversion we conjugate.



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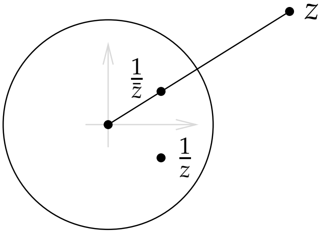

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LFTs take the set of straight lines and circles to the set of straight lines and circles.

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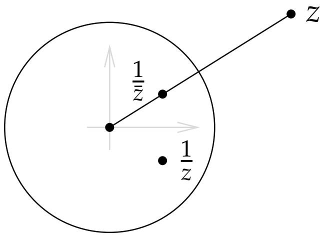

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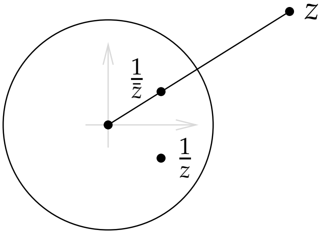

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Remark: A straight line is really a very large circle through ∞.