Cultivating Complex Analysis: Hurwitz's theorem (5.4.3)

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Theorem (Hurwitz)

Let $U \subset \mathbb{C}$ be open and $f_n : U \to \mathbb{C}$ a sequence of holomorphic functions converging uniformly on compact subsets to a holomorphic $f : U \to \mathbb{C}$. Suppose Γ is a cycle in U homologous to zero in U, such that $n(\Gamma; z)$ is 0 or 1 for all $z \notin \Gamma$. Suppose f has no zeros on Γ and k zeros (counting multiplicity) in $V = \{z \in U \setminus \Gamma : n(\Gamma; z) = 1\}$. Then there is an N such that for all $n \geq N$, f_n has k zeros (counting multiplicity) in V.

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"No zeros on Γ" is necessary: $\quad \Gamma = \partial\mathbb{D}, \quad f(z) = z - 1, \quad f_n(z) = z + (1 - \frac{1}{n})$.

Example: For every integer $k > 0$, $\exists\, N$ such that $\forall\, d \geq N$,

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Proof:

P_d are the partial sums of the power series of $\cos(z)$, which has exactly $2k$ zeros in $\Delta_{\pi k}(0)$.

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For any $\epsilon > 0$, $z^2 + \frac{1}{n}$ has two zeros in $\Delta_\epsilon(0)$, for large enough n: $\quad \pm i\sqrt{\frac{1}{n}}$.

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Corollary

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$\Rightarrow f_n$ not univalent. $\square$

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