Cultivating Complex Analysis: Harmonic functions Dirichlet problem in a disc and the Poisson kernel (7.2.1)

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On an unbounded domain, solution need not be unique.

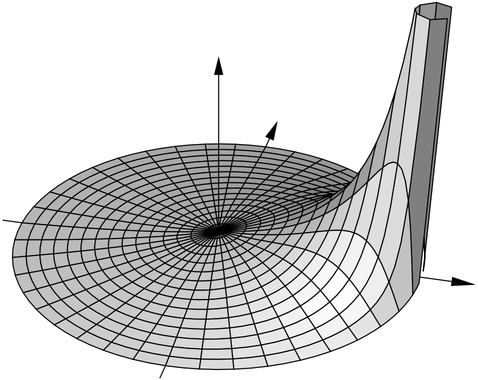
Exercise: The Dirichlet problem does not have a unique solution on the upper half plane $\mathbb{H}$.

The Poisson kernel for the unit disc $\mathbb{D} \subset \mathbb{C}$ is

$$P_r(\theta) = \frac{1}{2\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos\theta} = \frac{1}{2\pi} \operatorname{Re}\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right), \qquad \text{for } 0 \leq r < 1.$$

As a function of $z = re^{i\theta}$:

$$\frac{1}{2\pi} \operatorname{Re}\left(\frac{1+z}{1-z}\right)$$



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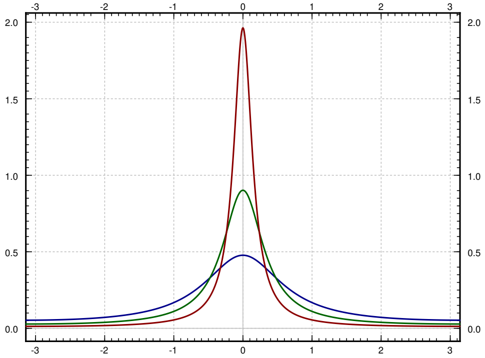
The graph of P_r as a function of θ on $[-\pi, \pi]$ for $r = 0.5$, $r = 0.7$, and $r = 0.85$:



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$$Pf(re^{i\theta}) = \begin{cases} \int_{-\pi}^{\pi} f(e^{it}) P_r(\theta - t)\, dt & \text{if } r < 1, \\ f(e^{i\theta}) & \text{if } r = 1, \end{cases}$$

is harmonic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$.

Proof: Let $z = re^{i\theta}$. For fixed t,

$$P_r(\theta - t) = \frac{1}{2\pi} \operatorname{Re}\left(\frac{1 + re^{i(\theta-t)}}{1 - re^{i(\theta-t)}}\right) = \frac{1}{2\pi} \operatorname{Re}\left(\frac{1 + ze^{-it}}{1 - ze^{-it}}\right)$$

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The integral from δ to π is exactly the same.

$$\left|\int_{-\delta}^{\delta}\left(f\left(e^{i(\theta-t)}\right)-f\left(e^{i\theta}\right)\right)P_r(t)\,dt\right|$$

$$\left|\int_{-\delta}^{\delta}\Big(f\big(e^{i(\theta-t)}\big)-f(e^{i\theta})\Big)P_r(t)\,dt\right| \le \int_{-\delta}^{\delta}\left|f\big(e^{i(\theta-t)}\big)-f(e^{i\theta})\right|P_r(t)\,dt$$

$$\left|\int_{-\delta}^{\delta}\left(f\left(e^{i(\theta-t)}\right)-f(e^{i\theta})\right)P_r(t)\,dt\right| \le \int_{-\delta}^{\delta}\left|f\left(e^{i(\theta-t)}\right)-f(e^{i\theta})\right|P_r(t)\,dt \le \int_{-\delta}^{\delta}\frac{\epsilon}{2}P_r(t)\,dt$$

$$\le \int_{-\pi}^{\pi}\frac{\epsilon}{2}P_r(t)\,dt$$

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$$\leq \int_{-\pi}^{\pi}\frac{\epsilon}{2}P_r(t)\,dt = \frac{\epsilon}{2}.$$

$$\left|\int_{-\delta}^{\delta}\Big(f\big(e^{i(\theta-t)}\big)-f(e^{i\theta})\Big)P_r(t)\,dt\right| \le \int_{-\delta}^{\delta}\Big|f\big(e^{i(\theta-t)}\big)-f(e^{i\theta})\Big|P_r(t)\,dt \le \int_{-\delta}^{\delta}\frac{\epsilon}{2}P_r(t)\,dt$$

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So $1-\delta' < r < 1 \quad \Rightarrow \quad |Pf(re^{i\theta})-f(e^{i\theta})|$

$$\left|\int_{-\delta}^{\delta}\Big(f\big(e^{i(\theta-t)}\big)-f(e^{i\theta})\Big)P_r(t)\,dt\right| \le \int_{-\delta}^{\delta}\left|f\big(e^{i(\theta-t)}\big)-f(e^{i\theta})\right|P_r(t)\,dt \le \int_{-\delta}^{\delta}\frac{\epsilon}{2}P_r(t)\,dt$$

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So $1-\delta' < r < 1 \quad \Rightarrow \quad |Pf(re^{i\theta})-f(e^{i\theta})| < \epsilon/4 + \epsilon/2 + \epsilon/4 = \epsilon.$

$$\left|\int_{-\delta}^{\delta}\Big(f\big(e^{i(\theta-t)}\big)-f(e^{i\theta})\Big)P_r(t)\,dt\right| \le \int_{-\delta}^{\delta}\Big|f\big(e^{i(\theta-t)}\big)-f(e^{i\theta})\Big|P_r(t)\,dt \le \int_{-\delta}^{\delta}\frac{\epsilon}{2}P_r(t)\,dt$$
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$\Rightarrow \quad Pf(re^{i\theta}) \to f(e^{i\theta})$ uniformly in θ as $r \uparrow 1$.

$$
\begin{aligned}
\left|\int_{-\delta}^{\delta}\Big(f\big(e^{i(\theta-t)}\big)-f(e^{i\theta})\Big)P_r(t)\,dt\right| &\le \int_{-\delta}^{\delta}\Big|f\big(e^{i(\theta-t)}\big)-f(e^{i\theta})\Big|\,P_r(t)\,dt \le \int_{-\delta}^{\delta}\frac{\epsilon}{2}P_r(t)\,dt \\
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Fix $z_0 = e^{i\theta_0} \in \partial\mathbb{D}$ and $\epsilon > 0$.

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Fix $z_0 = e^{i\theta_0} \in \partial\mathbb{D}$ and $\epsilon > 0$. $\quad f = Pf|_{\partial\mathbb{D}}$ is continuous, so $\exists\, \delta > 0$ such that

$|\theta - \theta_0| < \delta \quad \Rightarrow \quad |Pf(e^{i\theta}) - Pf(e^{i\theta_0})| < \epsilon/2$

$$\left|\int_{-\delta}^{\delta}\left(f\left(e^{i(\theta-t)}\right)-f(e^{i\theta})\right)P_r(t)\,dt\right| \leq \int_{-\delta}^{\delta}\left|f\left(e^{i(\theta-t)}\right)-f(e^{i\theta})\right|P_r(t)\,dt \leq \int_{-\delta}^{\delta}\frac{\epsilon}{2}P_r(t)\,dt$$

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$|\theta - \theta_0| < \delta \quad \Rightarrow \quad |Pf(e^{i\theta}) - Pf(e^{i\theta_0})| < {}^{\epsilon}/_{2}$

For possibly smaller δ

$1 - \delta < r \leq 1 \quad \Rightarrow \quad |Pf(re^{i\theta}) - Pf(e^{i\theta})| < {}^{\epsilon}/_{2} \quad$ for all θ.

$$\left|\int_{-\delta}^{\delta}\Big(f\big(e^{i(\theta-t)}\big)-f(e^{i\theta})\Big)P_r(t)\,dt\right| \le \int_{-\delta}^{\delta}\Big|f\big(e^{i(\theta-t)}\big)-f(e^{i\theta})\Big|\,P_r(t)\,dt \le \int_{-\delta}^{\delta}\frac{\epsilon}{2}P_r(t)\,dt$$

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For possibly smaller δ

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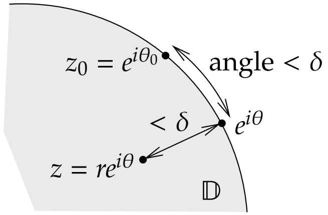

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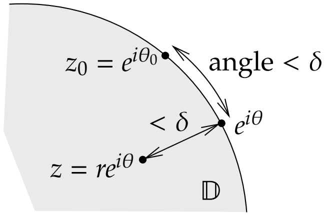

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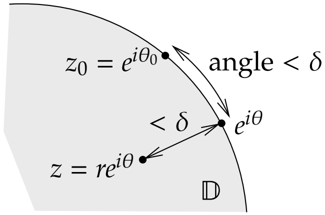

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Remark: We used polar coordinates (homeomorphism near z_0).

Translation and scaling gives (proof an exercise):

Corollary

Let $f: \partial\Delta_R(p) \to \mathbb{R}$ be continuous. Then $Pf: \overline{\Delta_R(p)} \to \mathbb{R}$, defined by

$$Pf(p + re^{i\theta}) = \begin{cases} \int_{-\pi}^{\pi} f(p + Re^{it}) P_{r/R}(\theta - t)\, dt & \text{if } r < R, \\ f(p + Re^{i\theta}) & \text{if } r = R, \end{cases}$$

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However, the Poisson kernel above is specific to the disk.

A quick consequence: If f is harmonic on a neighborhood of $\overline{\Delta_R(p)}$, then

$$f(p) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(p + Re^{it})\, dt.$$

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For general functions Cf and Pf are different in the disc:

Consider $f(z) = z + \bar{z} = 2\,\mathrm{Re}(z)$ on the unit circle.

$Pf(z) = z + \bar{z} \qquad Cf(z) = z$

Exercise: Given a bounded continuous $f\colon \mathbb{R} \to \mathbb{R}$, prove that $Pf\colon \overline{\mathbb{H}} \to \mathbb{R}$,

$$Pf(x+iy) = \begin{cases} \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{y}{(x-t)^2+y^2}\,dt & \text{if } y>0, \\ f(x) & \text{if } y=0, \end{cases}$$

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Exercise: Derive the *Schwarz integral formula*, which recovers a holomorphic function out of the real parts of the boundary values and the value of the imaginary part at one point. If $f\colon \overline{\mathbb{D}} \to \mathbb{C}$ is continuous and holomorphic on $\mathbb{D}$, then for all $z \in \mathbb{D}$,

$$f(z) = \frac{1}{2\pi i} \int_{\partial \mathbb{D}} \frac{\zeta + z}{\zeta - z} \frac{\operatorname{Re} f(\zeta)}{\zeta}\, d\zeta + i \operatorname{Im} f(0).$$