Cultivating Complex Analysis: Cauchy–Goursat, the “Cauchy for triangles” (3.2.2)

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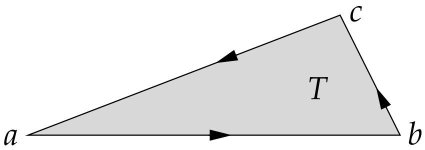

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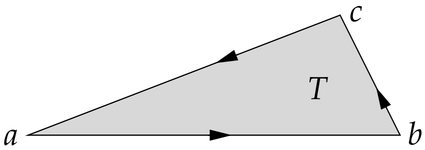
The boundary ∂T of T is defined as the cycle

$\partial T = [a, b] + [b, c] + [c, a]$.



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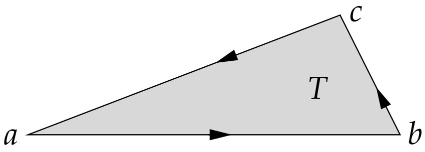

Note that our triangle T is the **solid** triangle (includes the interior).

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Suppose $U \subset \mathbb{C}$ is open, $f : U \to \mathbb{C}$ is holomorphic, and $T \subset U$ is a triangle. Then

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Suppose f is continuous and suppose $\exists\, T \subset U$ such that

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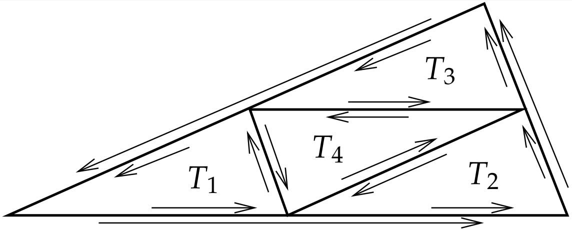
We will find a point where f is not complex differentiable.

Cut T into four subtriangles T_1, T_2, T_3, T_4
(cut each side in half)



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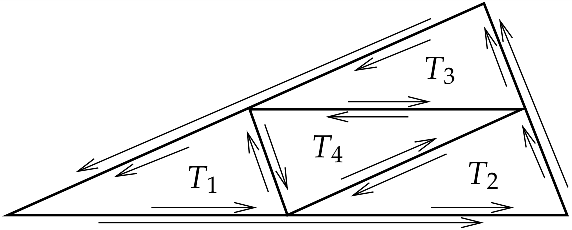
Orient each T_j positively:

The inner sides cancel.



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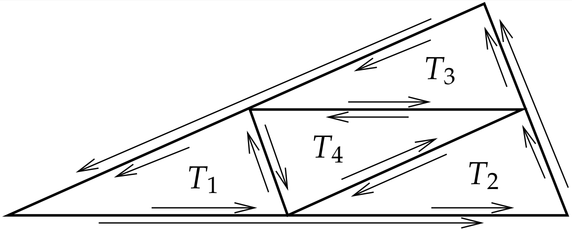

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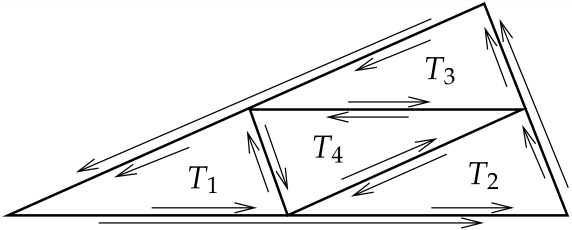

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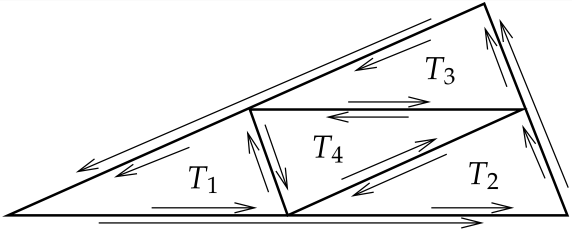

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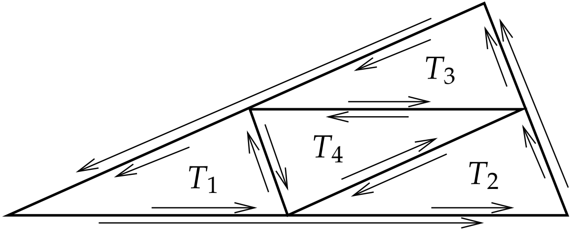

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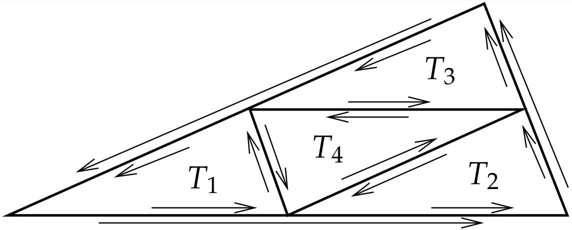

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Cut T^1 into subtriangles $T^1_1, T^1_2, T^1_3, T^1_4$. Integral over some ∂T^1_j is at least $\frac{c}{4^2}$, so label it T^2.

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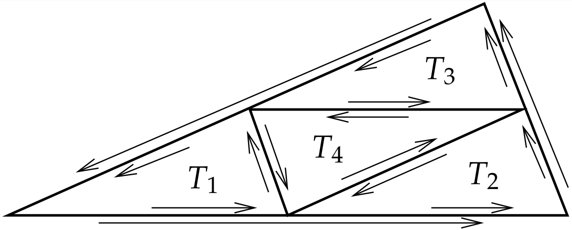

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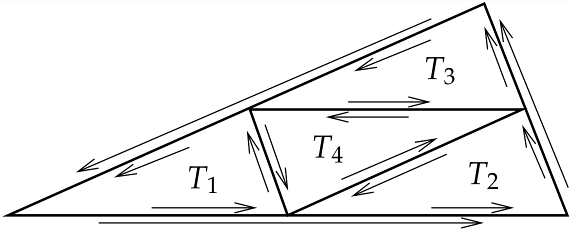

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Rinse and repeat.

After k iterations for the k^{th} triangle T^k, $\quad \left| \int_{\partial T^k} f(z)\,dz \right| \geq \frac{c}{4^k}.$

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Fix α. If $g(z_0) \neq 0$, we are done.

So assume $g(z_0) = 0$. Cauchy's theorem for polynomials says

$$\int_{\partial T^k} f(z)\,dz = \int_{\partial T^k} \big(f(z_0) + \alpha(z - z_0) + g(z)\big)\,dz = \int_{\partial T^k} g(z)\,dz.$$

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The length of ∂T^k is $\frac{\ell}{2^k}$, by the mean value theorem for integrals, $\exists\, z_k \in \partial T^k$ such that

$$|g(z_k)| = \frac{2^k}{\ell} \int_{\partial T^k} |g(z)|\, |dz|.$$

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Let $d = \operatorname{diam}(T)$. Then $|z_k - z_0| \leq \frac{d}{2^k}$ and

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$$\frac{c}{4^k} \leq \left| \int_{\partial T^k} f(z)\, dz \right| = \left| \int_{\partial T^k} g(z)\, dz \right| \leq \int_{\partial T^k} |g(z)|\, |dz|.$$

Let ℓ be the length of ∂T.

The length of ∂T^k is $\frac{\ell}{2^k}$, by the mean value theorem for integrals, $\exists\, z_k \in \partial T^k$ such that

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$z_k \neq z_0$ as $g(z_0) = 0$.

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So f is not complex differentiable at z_0. $\square$

A useful version of this result is the following exercise:

Exercise: Suppose $T \subset \mathbb{C}$ is a triangle and $f\colon T \to \mathbb{C}$ a continuous function whose restriction to the interior of T is holomorphic. Prove that $\int_{\partial T} f(z)\,dz = 0$.

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Hint: Passing some sort of limit under the integral is required.