

Cultivating Complex Analysis: The identity theorem (2.4.4)

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Common application:

If the function is zero on a nonempty open subset, then $f \equiv 0$.

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z^k is only zero at $0 \Rightarrow 0$ is an isolated zero of f .

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Z'_f is open and closed and U is connected \Rightarrow either $U = Z'_f$ or $Z'_f = \emptyset$.



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$$f(z) = (z - a)^k g(z),$$

where $g(z)$ is a power series at a such that $g(a) \neq 0$.