

Cultivating Complex Analysis: Homology versions of Cauchy (4.2)

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Definition

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Theorem (Cauchy integral formula (homology version))

Suppose $U \subset \mathbb{C}$ is open, $f: U \rightarrow \mathbb{C}$ is holomorphic, and Γ is a cycle in U homologous to zero in U . Then for $z \in U \setminus \Gamma$,

$$n(\Gamma; z)f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Proof: Define $g: U \times U \rightarrow \mathbb{C}$ by

$$g(\zeta, z) = \begin{cases} \frac{f(\zeta) - f(z)}{\zeta - z} & \text{if } \zeta \neq z, \\ f'(\zeta) & \text{if } \zeta = z. \end{cases}$$

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Let M be such that $|f(\zeta)| \leq M$ for $\zeta \in \Gamma$ and ℓ be the length of Γ .

$$|h(z)| = \left| \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta \right| \leq \int_{\Gamma} \left| \frac{f(\zeta)}{\zeta - z} \right| |d\zeta| \leq \frac{M\ell}{d(z, \Gamma)} \quad (d(z, \Gamma) \text{ is the distance of } z \text{ and } \Gamma).$$

$$z \rightarrow \infty \quad \Rightarrow \quad d(z, \Gamma) \rightarrow \infty \quad \Rightarrow \quad h(z) \rightarrow 0$$

So h is bounded, by Liouville h is constant, and the constant is zero.

Suppose $z \in U \setminus \Gamma$. Then

$$0 = h(z) = \int_{\Gamma} \frac{f(\zeta) - f(z)}{\zeta - z} d\zeta = \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta - f(z) (2\pi i) n(\Gamma; z). \quad \square$$

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Suppose $U \subset \mathbb{C}$ is open, $f: U \rightarrow \mathbb{C}$ is holomorphic, and Γ is a cycle in U homologous to zero in U . Then

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Remark: Cauchy integral formula and Cauchy's theorem are equivalent logically (if you prove one the other follows).

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Corollary

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Remark: Being “homologous” is an equivalence relation and the set of equivalence classes of cycles is an abelian group (under the cycle addition, exercise). This group is called the *first homology group* of U .