

Cultivating Complex Analysis: The open mapping theorem (5.5)

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For a holomorphic map, $f(V)$ is open whenever V is, unless f is constant.

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Take $w \in \Delta_\delta(f(p))$. For all $z \in \partial\Delta_r(p)$,

$$|(f(z) - w) - (f(z) - f(p))| = |f(p) - w| < \delta < |f(z) - f(p)|.$$

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By Rouché, $z \mapsto f(z) - w$ has at least one zero in $\Delta_r(p)$. So

$$\Delta_\delta(f(p)) \subset f(\Delta_r(p)). \quad \square$$

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The proof gives the more explicit:

$|f(z) - f(p)| > \delta$ for $z \in \partial\Delta_r(p)$, then $\Delta_\delta(f(p)) \subset f(\Delta_r(p))$.

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Typical application/exercise of the open mapping theorem is something like:

Exercise: Let $U \subset \mathbb{C}$ be a domain and let $f: U \rightarrow \mathbb{C}$ be holomorphic. Prove that if $(\operatorname{Im} f(z))^2 - (\operatorname{Re} f(z))^2 = 1$ for all $z \in U$, then f is constant.