

# Cultivating Complex Analysis: Inverse function theorem and automorphisms (2.2.3)

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**Example:** For any  $a, b \in \mathbb{C}$ ,  $a \neq 0$ , the function  $az + b$  is an automorphism of  $\mathbb{C}$ .

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Note that we also found that a holomorphic  $f$  preserves orientation (positive  $\det Df$ ).

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(Although we don't yet have enough machinery to prove these statements.)