Cultivating Complex Analysis: The geometry and topology of the plane (1.1.2) Complex-valued functions (1.1.3)

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Proposition

Complex addition, multiplication, division, and conjugation are continuous: Suppose $\{a_n\}$ and $\{b_n\}$ are two convergent sequences of complex numbers. Then,

(i) $\lim_{n\to\infty}(a_n + b_n) = \left(\lim_{n\to\infty} a_n\right) + \left(\lim_{n\to\infty} b_n\right)$,

(ii) $\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$,

(iii) $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{\lim_{n\to\infty} a_n}$, *as long as* $\lim_{n\to\infty} a_n \neq 0$,

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All these operations are defined in terms of operations on the real an imaginary parts which are continuous. Details left as exercise.

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Definition

An open and connected set $U \subset \mathbb{C}$ is called a *domain*.

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$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}.$$

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If $f\colon [a,b] \longrightarrow \mathbb{C}$, f is (Riemann) integrable if u and v are, and

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Proposition

Suppose $f : [a, b] \to \mathbb{C}$ is (Riemann) integrable. Then $|f|$ is (Riemann) integrable and

$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt.$$