

Cultivating Complex Analysis: Linear fractional transformations (1.4 part 2)

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Define the bijection $\Psi: \mathbb{C}_\infty \rightarrow \mathbb{CP}^1$ as

$$\Psi(z) = \begin{cases} [z : 1] & \text{if } z \in \mathbb{C}, \\ [1 : 0] & \text{if } z = \infty. \end{cases}$$

We claim an LFT corresponds to an invertible 2×2 matrix:

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As $M \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} az+bw \\ cz+dw \end{bmatrix}$, the function f corresponds to the linear map $v \mapsto Mv$ on \mathbb{C}^2 .

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We have the commutative diagram:

$$\begin{array}{ccc}
 \mathbb{C}^2 \setminus \{0\} & \xrightarrow{M} & \mathbb{C}^2 \setminus \{0\} \\
 \downarrow \pi & & \downarrow \pi \\
 \mathbb{CP}^1 & \xrightarrow{\Psi \circ f \circ \Psi^{-1}} & \mathbb{CP}^1 \\
 \Psi \updownarrow \Psi^{-1} & & \Psi \updownarrow \Psi^{-1} \\
 \mathbb{C}_\infty & \xrightarrow{f} & \mathbb{C}_\infty
 \end{array}$$

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The group is generated by T_a , D_a , and I for $a \in \mathbb{C}$.

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Why? $C(z)$ is in the unit disc if

$$1 > \left| \frac{z - i}{z + i} \right| = \frac{|z - i|}{|z + i|}, \quad \text{in other words if} \quad |z + i| > |z - i|.$$

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