Cultivating Complex Analysis: Wild world of essential singularities, Casorati–Weierstrass (5.2.3)

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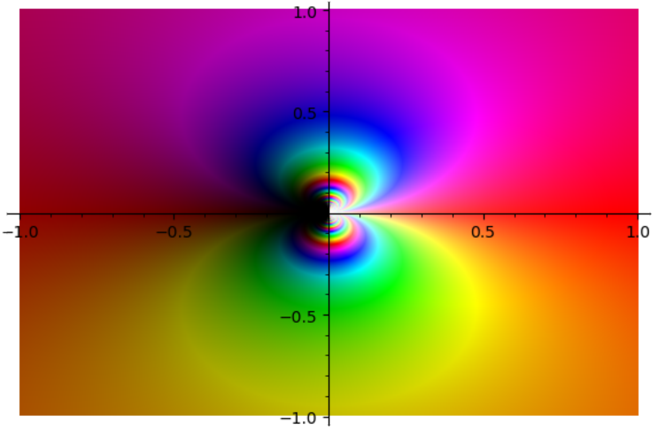
Values of $e^{1/z}$ are shown using modulus as brightness, argument as hue (color).



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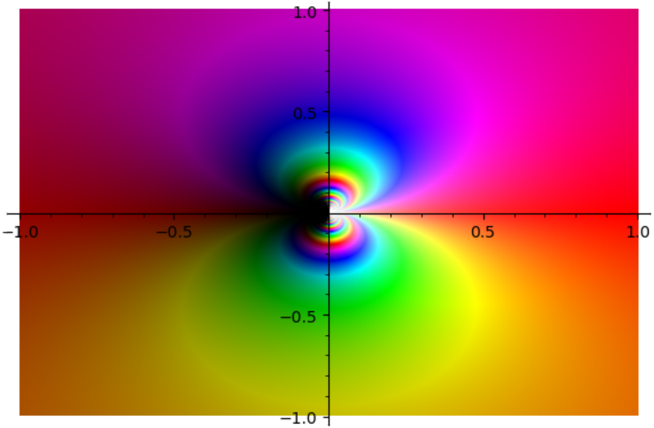

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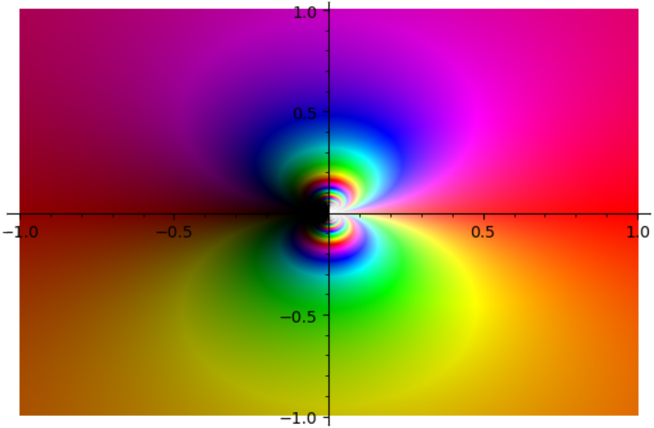
We actually get $\mathbb{C} \setminus \{0\}$ as
the image of any neighborhood of 0.



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$$f\big(\Delta_r(p) \setminus \{p\}\big) = \big\{w \in \mathbb{C} : w = f(z),\, z \in \Delta_r(p) \setminus \{p\}\big\} \quad \textit{is dense in } \mathbb{C}.$$

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Picard's theorem says
$f\left(\Delta_r(p) \setminus \{p\}\right) = \mathbb{C}$ or
$f\left(\Delta_r(p) \setminus \{p\}\right) = \mathbb{C} \setminus \{z_0\}$ for some z_0.

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Suppose $f \colon U \setminus \{p\} \to \mathbb{C}$ is holomorphic, $\Delta_r(p) \subset U$, and $\Delta_s(q)$ is such that

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If $g(p) \neq 0$, $\quad \Rightarrow \quad f$ has a removable singularity at p.

In either case, f does not have an essential singularity at p. $\square$

Exercise: Prove the converse of Casorati–Weierstrass. Let $U \subset \mathbb{C}$ be open, $p \in U$, and $f \colon U \setminus \{p\} \to \mathbb{C}$ holomorphic. Prove that if $f\big(\Delta_r(p) \setminus \{p\}\big)$ is dense in $\mathbb{C}$ for all $r > 0$ such that $\Delta_r(p) \subset U$, then f has an essential singularity at p.

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Exercise: Suppose that $g: \Delta_r(p) \setminus \{p\} \to \mathbb{C}$ has an isolated singularity. Prove $f(z) = e^{g(z)}$ has either a removable or an essential singularity.

Exercise: Suppose $f: \mathbb{C} \to \mathbb{C}$ is holomorphic and nonconstant. Prove $f(\mathbb{C})$ is dense in $\mathbb{C}$. Remark: The so-called "little Picard theorem" says that $f(\mathbb{C})$ is actually everything minus possibly one point, but that is much harder to prove.