Cultivating Complex Analysis: Schwarz’s lemma (3.5.1)

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b) We'll see later that every domain "without holes" (except $\mathbb{C}$) is biholomorphic to $\mathbb{D}$, so it tells us about global behavior as well.

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As $g(0) = f'(0)$, the same conclusion holds if $|f'(0)| = 1$. $\square$

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Schwarz's lemma says all holomorphic functions behave this way, not just z^n.

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For (ii) use Cauchy estimates on the first nonzero nonlinear term of f^ℓ.