Cultivating Complex Analysis: Power series (2.3 part 3)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

Let $K \subset \mathbb{C}$ be a set. A power series $\sum c_n(z-p)^n$ *converges uniformly absolutely* for $z \in K$ when $\sum |c_n||z-p|^n$ converges uniformly for $z \in K$.

Let $K \subset \mathbb{C}$ be a set. A power series $\sum c_n(z-p)^n$ *converges uniformly absolutely* for $z \in K$ when $\sum |c_n||z-p|^n$ converges uniformly for $z \in K$.

Suppose a series converges uniformly absolutely.

Let $K \subset \mathbb{C}$ be a set. A power series $\sum c_n(z-p)^n$ *converges uniformly absolutely* for $z \in K$ when $\sum |c_n||z-p|^n$ converges uniformly for $z \in K$.

Suppose a series converges uniformly absolutely.

It converges absolutely, so it converges,

Let $K \subset \mathbb{C}$ be a set. A power series $\sum c_n(z-p)^n$ *converges uniformly absolutely* for $z \in K$ when $\sum |c_n||z-p|^n$ converges uniformly for $z \in K$.

Suppose a series converges uniformly absolutely.

It converges absolutely, so it converges, and

$$\left|\sum_{n=0}^{\infty} c_n(z-p)^n - \sum_{n=0}^{m} c_n(z-p)^n\right| = \left|\sum_{n=m+1}^{\infty} c_n(z-p)^n\right| \leq \sum_{n=m+1}^{\infty} |c_n||z-p|^n.$$

Let $K \subset \mathbb{C}$ be a set. A power series $\sum c_n(z-p)^n$ *converges uniformly absolutely* for $z \in K$ when $\sum |c_n||z-p|^n$ converges uniformly for $z \in K$.

Suppose a series converges uniformly absolutely.

It converges absolutely, so it converges, and

$$\left|\sum_{n=0}^{\infty} c_n(z-p)^n - \sum_{n=0}^{m} c_n(z-p)^n\right| = \left|\sum_{n=m+1}^{\infty} c_n(z-p)^n\right| \leq \sum_{n=m+1}^{\infty} |c_n||z-p|^n.$$

The RHS $\to 0$ uniformly in $z \in K$ as $m \to \infty$.

Let $K \subset \mathbb{C}$ be a set. A power series $\sum c_n(z-p)^n$ *converges uniformly absolutely* for $z \in K$ when $\sum |c_n||z-p|^n$ converges uniformly for $z \in K$.

Suppose a series converges uniformly absolutely.

It converges absolutely, so it converges, and

$$\left|\sum_{n=0}^{\infty} c_n(z-p)^n - \sum_{n=0}^{m} c_n(z-p)^n\right| = \left|\sum_{n=m+1}^{\infty} c_n(z-p)^n\right| \leq \sum_{n=m+1}^{\infty} |c_n||z-p|^n.$$

The RHS $\to 0$ uniformly in $z \in K$ as $m \to \infty$.

So a uniformly absolutely convergent series converges uniformly.

Proposition

Let $\sum c_n(z-p)^n$ be a power series with radius of convergence $R > 0$. If $0 < r < R$, then the power series converges uniformly absolutely in $\overline{\Delta_r(p)}$.

Proposition

Let $\sum c_n(z-p)^n$ be a power series with radius of convergence $R > 0$. If $0 < r < R$, then the power series converges uniformly absolutely in $\overline{\Delta_r(p)}$. Furthermore, let $U = \Delta_R(p)$ if $R < \infty$ and $U = \mathbb{C}$ if $R = \infty$, and let $K \subset U$ be compact. Then the series converges uniformly absolutely on K.

Proposition

Let $\sum c_n(z-p)^n$ be a power series with radius of convergence $R > 0$. If $0 < r < R$, then the power series converges uniformly absolutely in $\overline{\Delta_r(p)}$. Furthermore, let $U = \Delta_R(p)$ if $R < \infty$ and $U = \mathbb{C}$ if $R = \infty$, and let $K \subset U$ be compact. Then the series converges uniformly absolutely on K.

Proof: WLOG suppose $R < \infty$ and let $0 < r < R$.

Proposition

Let $\sum c_n(z-p)^n$ be a power series with radius of convergence $R > 0$. If $0 < r < R$, then the power series converges uniformly absolutely in $\overline{\Delta_r(p)}$. Furthermore, let $U = \Delta_R(p)$ if $R < \infty$ and $U = \mathbb{C}$ if $R = \infty$, and let $K \subset U$ be compact. Then the series converges uniformly absolutely on K.

Proof: WLOG suppose $R < \infty$ and let $0 < r < R$.

$\sum c_n(z-p)^n$ converges absolutely on $\Delta_R(p) \quad \Rightarrow \quad \sum |c_n| r^n$ converges (and so does any tail).

Proposition

Let $\sum c_n(z-p)^n$ be a power series with radius of convergence $R > 0$. If $0 < r < R$, then the power series converges uniformly absolutely in $\overline{\Delta_r(p)}$. Furthermore, let $U = \Delta_R(p)$ if $R < \infty$ and $U = \mathbb{C}$ if $R = \infty$, and let $K \subset U$ be compact. Then the series converges uniformly absolutely on K.

Proof: WLOG suppose $R < \infty$ and let $0 < r < R$.

$\sum c_n(z-p)^n$ converges absolutely on $\Delta_R(p)$ $\quad\Rightarrow\quad$ $\sum |c_n| r^n$ converges (and so does any tail).

So for $z \in \overline{\Delta_r(p)}$,

$$\left| \sum_{n=0}^{\infty} |c_n||z-p|^n - \sum_{n=0}^{m} |c_n||z-p|^n \right| \le \sum_{n=m+1}^{\infty} |c_n| r^n.$$

Proposition

Let $\sum c_n(z-p)^n$ be a power series with radius of convergence $R > 0$. If $0 < r < R$, then the power series converges uniformly absolutely in $\overline{\Delta_r(p)}$. Furthermore, let $U = \Delta_R(p)$ if $R < \infty$ and $U = \mathbb{C}$ if $R = \infty$, and let $K \subset U$ be compact. Then the series converges uniformly absolutely on K.

Proof: WLOG suppose $R < \infty$ and let $0 < r < R$.

$\sum c_n(z-p)^n$ converges absolutely on $\Delta_R(p)$ $\quad\Rightarrow\quad$ $\sum |c_n| r^n$ converges (and so does any tail).

So for $z \in \overline{\Delta_r(p)}$,

$$\left| \sum_{n=0}^{\infty} |c_n||z-p|^n - \sum_{n=0}^{m} |c_n||z-p|^n \right| \leq \sum_{n=m+1}^{\infty} |c_n| r^n.$$

The RHS, which does not depend on z, goes to zero as $m \to \infty$.

Proposition

Let $\sum c_n(z-p)^n$ be a power series with radius of convergence $R > 0$. If $0 < r < R$, then the power series converges uniformly absolutely in $\overline{\Delta_r(p)}$. Furthermore, let $U = \Delta_R(p)$ if $R < \infty$ and $U = \mathbb{C}$ if $R = \infty$, and let $K \subset U$ be compact. Then the series converges uniformly absolutely on K.

Proof: WLOG suppose $R < \infty$ and let $0 < r < R$.

$\sum c_n(z-p)^n$ converges absolutely on $\Delta_R(p)$ $\quad\Rightarrow\quad$ $\sum |c_n| r^n$ converges (and so does any tail).

So for $z \in \overline{\Delta_r(p)}$,

$$\left| \sum_{n=0}^{\infty} |c_n||z-p|^n - \sum_{n=0}^{m} |c_n||z-p|^n \right| \leq \sum_{n=m+1}^{\infty} |c_n| r^n.$$

The RHS, which does not depend on z, goes to zero as $m \to \infty$.
Hence $\sum |c_n||z-p|^n$ converges uniformly in $\overline{\Delta_r(p)}$.

Proposition

Let $\sum c_n(z-p)^n$ be a power series with radius of convergence $R > 0$. If $0 < r < R$, then the power series converges uniformly absolutely in $\overline{\Delta_r(p)}$. Furthermore, let $U = \Delta_R(p)$ if $R < \infty$ and $U = \mathbb{C}$ if $R = \infty$, and let $K \subset U$ be compact. Then the series converges uniformly absolutely on K.

Proof: WLOG suppose $R < \infty$ and let $0 < r < R$.

$\sum c_n(z-p)^n$ converges absolutely on $\Delta_R(p)$ $\quad\Rightarrow\quad$ $\sum |c_n| r^n$ converges (and so does any tail).

So for $z \in \overline{\Delta_r(p)}$,

$$\left| \sum_{n=0}^{\infty} |c_n||z-p|^n - \sum_{n=0}^{m} |c_n||z-p|^n \right| \leq \sum_{n=m+1}^{\infty} |c_n| r^n.$$

The RHS, which does not depend on z, goes to zero as $m \to \infty$.
Hence $\sum |c_n||z-p|^n$ converges uniformly in $\overline{\Delta_r(p)}$.

If $K \subset \Delta_R(p)$ is compact, $\exists\, r < R$ such that $K \subset \Delta_r(p)$

Proposition

Let $\sum c_n(z-p)^n$ be a power series with radius of convergence $R > 0$. If $0 < r < R$, then the power series converges uniformly absolutely in $\overline{\Delta_r(p)}$. Furthermore, let $U = \Delta_R(p)$ if $R < \infty$ and $U = \mathbb{C}$ if $R = \infty$, and let $K \subset U$ be compact. Then the series converges uniformly absolutely on K.

Proof: WLOG suppose $R < \infty$ and let $0 < r < R$.

$\sum c_n(z-p)^n$ converges absolutely on $\Delta_R(p) \quad \Rightarrow \quad \sum |c_n| r^n$ converges (and so does any tail).

So for $z \in \overline{\Delta_r(p)}$,

$$\left| \sum_{n=0}^{\infty} |c_n||z-p|^n - \sum_{n=0}^{m} |c_n||z-p|^n \right| \leq \sum_{n=m+1}^{\infty} |c_n| r^n.$$

The RHS, which does not depend on z, goes to zero as $m \to \infty$.
Hence $\sum |c_n||z-p|^n$ converges uniformly in $\overline{\Delta_r(p)}$.

If $K \subset \Delta_R(p)$ is compact, $\exists\, r < R$ such that $K \subset \Delta_r(p)$
(consider an open cover of K by discs $\Delta_r(p)$ for all $r < R$).

Proposition

Let $\sum c_n(z-p)^n$ be a power series with radius of convergence $R > 0$. If $0 < r < R$, then the power series converges uniformly absolutely in $\overline{\Delta_r(p)}$. Furthermore, let $U = \Delta_R(p)$ if $R < \infty$ and $U = \mathbb{C}$ if $R = \infty$, and let $K \subset U$ be compact. Then the series converges uniformly absolutely on K.

Proof: WLOG suppose $R < \infty$ and let $0 < r < R$.

$\sum c_n(z-p)^n$ converges absolutely on $\Delta_R(p)$ $\quad\Rightarrow\quad$ $\sum |c_n| r^n$ converges (and so does any tail).

So for $z \in \overline{\Delta_r(p)}$,

$$\left| \sum_{n=0}^{\infty} |c_n||z-p|^n - \sum_{n=0}^{m} |c_n||z-p|^n \right| \leq \sum_{n=m+1}^{\infty} |c_n| r^n.$$

The RHS, which does not depend on z, goes to zero as $m \to \infty$.
Hence $\sum |c_n||z-p|^n$ converges uniformly in $\overline{\Delta_r(p)}$.

If $K \subset \Delta_R(p)$ is compact, $\exists\, r < R$ such that $K \subset \Delta_r(p)$
(consider an open cover of K by discs $\Delta_r(p)$ for all $r < R$).

The result follows. $\square$

Let us mention a couple of useful results as exercises.

Let us mention a couple of useful results as exercises.

Exercise: (Weierstrass M-test) Let X be a set and $f_n : X \to \mathbb{C}$ is a sequence of functions such that $|f_n(x)| \leq M_n$ for all $x \in X$ and $n \in \mathbb{N}$.

Let us mention a couple of useful results as exercises.

Exercise: (Weierstrass M-test) Let X be a set and $f_n : X \to \mathbb{C}$ is a sequence of functions such that $|f_n(x)| \leq M_n$ for all $x \in X$ and $n \in \mathbb{N}$.

If $\sum M_n < \infty$, then $\sum f_n(x)$ converges uniformly absolutely on X.

Let us mention a couple of useful results as exercises.

Exercise: (Weierstrass M-test) Let X be a set and $f_n \colon X \to \mathbb{C}$ is a sequence of functions such that $|f_n(x)| \le M_n$ for all $x \in X$ and $n \in \mathbb{N}$.

If $\sum M_n < \infty$, then $\sum f_n(x)$ converges uniformly absolutely on X.

Exercise: Suppose $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ have a radius of convergence at least $r > 0$. Show that $\sum_{n=0}^{\infty} (a_n + b_n) z^n$ has a radius of convergence at least r and converges to the sum of the two series.