Cultivating Complex Analysis: Zeros of holomorphic functions (5.1)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

Recall, a *zero* of a function f is a point z such that $f(z) = 0$.

Recall, a *zero* of a function f is a point z such that $f(z) = 0$.

By the identity theorem, zeros of (nonconstant) holomorphic functions are isolated.

Recall, a *zero* of a function f is a point z such that $f(z) = 0$.

By the identity theorem, zeros of (nonconstant) holomorphic functions are isolated.

Lemma

Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ be holomorphic, $p \in U$, and f has an isolated zero at p.

Recall, a *zero* of a function f is a point z such that $f(z) = 0$.

By the identity theorem, zeros of (nonconstant) holomorphic functions are isolated.

Lemma

Let $U \subset \mathbb{C}$ be open, $f: U \to \mathbb{C}$ be holomorphic, $p \in U$, and f has an isolated zero at p. Then there exists a unique $k \in \mathbb{N}$ and a holomorphic $g: U \to \mathbb{C}$ such that

$$f(z) = (z-p)^k g(z) \qquad \text{and} \qquad g(p) \neq 0.$$

Recall, a *zero* of a function f is a point z such that $f(z) = 0$.

By the identity theorem, zeros of (nonconstant) holomorphic functions are isolated.

Lemma

Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ be holomorphic, $p \in U$, and f has an isolated zero at p. Then there exists a unique $k \in \mathbb{N}$ and a holomorphic $g : U \to \mathbb{C}$ such that

$$f(z) = (z - p)^k g(z) \qquad \text{and} \qquad g(p) \neq 0.$$

Furthermore, k is the smallest integer such that the k^{th} derivative $f^{(k)}(p) \neq 0$.

Recall, a *zero* of a function f is a point z such that $f(z) = 0$.

By the identity theorem, zeros of (nonconstant) holomorphic functions are isolated.

Lemma

Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ be holomorphic, $p \in U$, and f has an isolated zero at p. Then there exists a unique $k \in \mathbb{N}$ and a holomorphic $g : U \to \mathbb{C}$ such that

$$f(z) = (z-p)^k g(z) \qquad \text{and} \qquad g(p) \neq 0.$$

Furthermore, k is the smallest integer such that the k^{th} derivative $f^{(k)}(p) \neq 0$.

Before we prove the lemma, let us give a name to this integer k.

Recall, a *zero* of a function f is a point z such that $f(z) = 0$.

By the identity theorem, zeros of (nonconstant) holomorphic functions are isolated.

Lemma

Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ be holomorphic, $p \in U$, and f has an isolated zero at p. Then there exists a unique $k \in \mathbb{N}$ and a holomorphic $g : U \to \mathbb{C}$ such that

$$f(z) = (z-p)^k g(z) \qquad \text{and} \qquad g(p) \neq 0.$$

Furthermore, k is the smallest integer such that the k^{th} derivative $f^{(k)}(p) \neq 0$.

Before we prove the lemma, let us give a name to this integer k.

Definition

The k from the lemma is called the *order* of the zero at p.

Recall, a *zero* of a function f is a point z such that $f(z) = 0$.

By the identity theorem, zeros of (nonconstant) holomorphic functions are isolated.

Lemma

Let $U \subset \mathbb{C}$ be open, $f : U \to \mathbb{C}$ be holomorphic, $p \in U$, and f has an isolated zero at p. Then there exists a unique $k \in \mathbb{N}$ and a holomorphic $g : U \to \mathbb{C}$ such that

$$f(z) = (z - p)^k g(z) \qquad \text{and} \qquad g(p) \neq 0.$$

Furthermore, k is the smallest integer such that the k^{th} derivative $f^{(k)}(p) \neq 0$.

Before we prove the lemma, let us give a name to this integer k.

Definition

The k from the lemma is called the *order* of the zero at p.
If the order is 1, we say p is a *simple zero*.

Recall, a *zero* of a function f is a point z such that $f(z) = 0$.

By the identity theorem, zeros of (nonconstant) holomorphic functions are isolated.

Lemma

Let $U \subset \mathbb{C}$ be open, $f: U \to \mathbb{C}$ be holomorphic, $p \in U$, and f has an isolated zero at p. Then there exists a unique $k \in \mathbb{N}$ and a holomorphic $g: U \to \mathbb{C}$ such that

$$f(z) = (z-p)^k g(z) \qquad \text{and} \qquad g(p) \neq 0.$$

Furthermore, k is the smallest integer such that the k^{th} derivative $f^{(k)}(p) \neq 0$.

Before we prove the lemma, let us give a name to this integer k.

Definition

The k from the lemma is called the *order* of the zero at p.
If the order is 1, we say p is a *simple zero*.
We also say that k is the *multiplicity* of the zero.

An equivalent definition of *order* is the largest k such that $\frac{f(z)}{(z-p)^k}$ is bounded near p.

An equivalent definition of *order* is the largest k such that $\frac{f(z)}{(z-p)^k}$ is bounded near p.

In fact (exercise), there is a disc $\Delta_r(p)$ and some $C_1 > 0$ and $C_2 > 0$ such that

$$C_1|z-p|^k \leq |f(z)| \leq C_2|z-p|^k \qquad \forall z \in \Delta_r(p).$$

An equivalent definition of *order* is the largest k such that $\frac{f(z)}{(z-p)^k}$ is bounded near p.

In fact (exercise), there is a disc $\Delta_r(p)$ and some $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 |z - p|^k \leq |f(z)| \leq C_2 |z - p|^k \qquad \forall z \in \Delta_r(p).$$

The conclusion of the lemma still holds if $f(p) \neq 0$, in which case order is $k = 0$ (and $g = f$).

An equivalent definition of *order* is the largest k such that $\frac{f(z)}{(z-p)^k}$ is bounded near p.

In fact (exercise), there is a disc $\Delta_r(p)$ and some $C_1 > 0$ and $C_2 > 0$ such that

$$C_1|z-p|^k \leq |f(z)| \leq C_2|z-p|^k \qquad \forall z \in \Delta_r(p).$$

The conclusion of the lemma still holds if $f(p) \neq 0$, in which case order is $k = 0$ (and $g = f$).

So we could say that if $f(p) \neq 0$, then f *has a zero of order* 0.

An equivalent definition of *order* is the largest k such that $\frac{f(z)}{(z-p)^k}$ is bounded near p.

In fact (exercise), there is a disc $\Delta_r(p)$ and some $C_1 > 0$ and $C_2 > 0$ such that

$$C_1|z-p|^k \leq |f(z)| \leq C_2|z-p|^k \qquad \forall z \in \Delta_r(p).$$

The conclusion of the lemma still holds if $f(p) \neq 0$, in which case order is $k = 0$ (and $g = f$).

So we could say that if $f(p) \neq 0$, then f *has a zero of order* 0.

We'll even see negative orders in just a bit.

An equivalent definition of *order* is the largest k such that $\frac{f(z)}{(z-p)^k}$ is bounded near p.

In fact (exercise), there is a disc $\Delta_r(p)$ and some $C_1 > 0$ and $C_2 > 0$ such that

$$C_1|z-p|^k \leq |f(z)| \leq C_2|z-p|^k \qquad \forall z \in \Delta_r(p).$$

The conclusion of the lemma still holds if $f(p) \neq 0$, in which case order is $k = 0$ (and $g = f$).

So we could say that if $f(p) \neq 0$, then f *has a zero of order* 0.

We'll even see negative orders in just a bit.

To avoid confusion, by "f has a zero," we mean an honest zero of positive order.

An equivalent definition of *order* is the largest k such that $\frac{f(z)}{(z-p)^k}$ is bounded near p.

In fact (exercise), there is a disc $\Delta_r(p)$ and some $C_1 > 0$ and $C_2 > 0$ such that

$$C_1|z-p|^k \leq |f(z)| \leq C_2|z-p|^k \qquad \forall z \in \Delta_r(p).$$

The conclusion of the lemma still holds if $f(p) \neq 0$, in which case order is $k = 0$ (and $g = f$).

So we could say that if $f(p) \neq 0$, then f *has a zero of order* 0.

We'll even see negative orders in just a bit.

To avoid confusion, by "f has a zero," we mean an honest zero of positive order.

If $f^{(k)}(p) = 0$ for all k, one could say that f has a zero of infinite order.

An equivalent definition of *order* is the largest k such that $\frac{f(z)}{(z-p)^k}$ is bounded near p.

In fact (exercise), there is a disc $\Delta_r(p)$ and some $C_1 > 0$ and $C_2 > 0$ such that

$$C_1|z-p|^k \leq |f(z)| \leq C_2|z-p|^k \qquad \forall z \in \Delta_r(p).$$

The conclusion of the lemma still holds if $f(p) \neq 0$, in which case order is $k = 0$ (and $g = f$).

So we could say that if $f(p) \neq 0$, then f *has a zero of order* 0.

We'll even see negative orders in just a bit.

To avoid confusion, by "f has a zero," we mean an honest zero of positive order.

If $f^{(k)}(p) = 0$ for all k, one could say that f has a zero of infinite order.

For f holomorphic, infinite order $\Rightarrow$ the power series is zero $\Rightarrow f$ is identically zero.

An equivalent definition of *order* is the largest k such that $\frac{f(z)}{(z-p)^k}$ is bounded near p.

In fact (exercise), there is a disc $\Delta_r(p)$ and some $C_1 > 0$ and $C_2 > 0$ such that

$$C_1|z-p|^k \leq |f(z)| \leq C_2|z-p|^k \qquad \forall z \in \Delta_r(p).$$

The conclusion of the lemma still holds if $f(p) \neq 0$, in which case order is $k = 0$ (and $g = f$).

So we could say that if $f(p) \neq 0$, then f *has a zero of order* 0.

We'll even see negative orders in just a bit.

To avoid confusion, by "f has a zero," we mean an honest zero of positive order.

If $f^{(k)}(p) = 0$ for all k, one could say that f has a zero of infinite order.

For f holomorphic, infinite order $\Rightarrow$ the power series is zero $\Rightarrow f$ is identically zero.

So every zero of a nonconstant holomorphic function has finite order.

An equivalent definition of *order* is the largest k such that $\frac{f(z)}{(z-p)^k}$ is bounded near p.

In fact (exercise), there is a disc $\Delta_r(p)$ and some $C_1 > 0$ and $C_2 > 0$ such that

$$C_1|z-p|^k \leq |f(z)| \leq C_2|z-p|^k \qquad \forall z \in \Delta_r(p).$$

The conclusion of the lemma still holds if $f(p) \neq 0$, in which case order is $k = 0$ (and $g = f$).

So we could say that if $f(p) \neq 0$, then *f has a zero of order* 0.

We'll even see negative orders in just a bit.

To avoid confusion, by "f has a zero," we mean an honest zero of positive order.

If $f^{(k)}(p) = 0$ for all k, one could say that f has a zero of infinite order.

For f holomorphic, infinite order $\Rightarrow$ the power series is zero $\Rightarrow f$ is identically zero.

So every zero of a nonconstant holomorphic function has finite order.

This is not true for just real differentiable (not holomorphic) functions (see the exercises):
E.g., let $f(0) = 0$ and $f(x) = e^{-1/x^2}$ for $x \in \mathbb{R} \setminus \{0\}$.
f is infinitely differentiable, $f^{(k)}(0) = 0$ for all k, but f has an isolated zero.

Proof of the lemma: On $U \setminus \{p\}$, $g(z) = \frac{f(z)}{(z-p)^k}$ is holomorphic for any k.

Proof of the lemma: On $U \setminus \{p\}$, $g(z) = \frac{f(z)}{(z-p)^k}$ is holomorphic for any k.

For z in some disc $\Delta_r(p)$,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-p)^n$$

Proof of the lemma: On $U \setminus \{p\}$, $g(z) = \frac{f(z)}{(z-p)^k}$ is holomorphic for any k.

For z in some disc $\Delta_r(p)$,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-p)^n = \sum_{n=k}^{\infty} c_n (z-p)^n$$

Proof of the lemma: On $U \setminus \{p\}$, $g(z) = \frac{f(z)}{(z-p)^k}$ is holomorphic for any k.

For z in some disc $\Delta_r(p)$,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-p)^n = \sum_{n=k}^{\infty} c_n (z-p)^n = (z-p)^k \sum_{n=0}^{\infty} c_{n+k} (z-p)^n,$$

Proof of the lemma: On $U \setminus \{p\}$, $g(z) = \frac{f(z)}{(z-p)^k}$ is holomorphic for any k.

For z in some disc $\Delta_r(p)$,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-p)^n = \sum_{n=k}^{\infty} c_n (z-p)^n = (z-p)^k \sum_{n=0}^{\infty} c_{n+k} (z-p)^n,$$

where k is the smallest n such that $c_n \neq 0$ (hence the "Furthermore").

Proof of the lemma: On $U \setminus \{p\}$, $g(z) = \frac{f(z)}{(z-p)^k}$ is holomorphic for any k.

For z in some disc $\Delta_r(p)$,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-p)^n = \sum_{n=k}^{\infty} c_n (z-p)^n = (z-p)^k \sum_{n=0}^{\infty} c_{n+k} (z-p)^n,$$

where k is the smallest n such that $c_n \neq 0$ (hence the "Furthermore").

Clearly $k > 0$.

Proof of the lemma: On $U \setminus \{p\}$, $g(z) = \frac{f(z)}{(z-p)^k}$ is holomorphic for any k.

For z in some disc $\Delta_r(p)$,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-p)^n = \sum_{n=k}^{\infty} c_n (z-p)^n = (z-p)^k \sum_{n=0}^{\infty} c_{n+k} (z-p)^n,$$

where k is the smallest n such that $c_n \neq 0$ (hence the “Furthermore”).

Clearly $k > 0$.

The series $\sum_{n=0}^{\infty} c_{n+k}(z-p)^n$ is equal to $\frac{f(z)}{(z-p)^k}$ on $\Delta_r(p) \setminus \{p\}$.

Proof of the lemma: On $U \setminus \{p\}$, $g(z) = \frac{f(z)}{(z-p)^k}$ is holomorphic for any k.

For z in some disc $\Delta_r(p)$,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-p)^n = \sum_{n=k}^{\infty} c_n (z-p)^n = (z-p)^k \sum_{n=0}^{\infty} c_{n+k} (z-p)^n,$$

where k is the smallest n such that $c_n \neq 0$ (hence the "Furthermore").

Clearly $k > 0$.

The series $\sum_{n=0}^{\infty} c_{n+k}(z-p)^n$ is equal to $\frac{f(z)}{(z-p)^k}$ on $\Delta_r(p) \setminus \{p\}$.

So this series gives g near p. Let $g(p) = c_k$, and $g\colon U \to \mathbb{C}$ is holomorphic.

Proof of the lemma: On $U \setminus \{p\}$, $g(z) = \frac{f(z)}{(z-p)^k}$ is holomorphic for any k.

For z in some disc $\Delta_r(p)$,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-p)^n = \sum_{n=k}^{\infty} c_n (z-p)^n = (z-p)^k \sum_{n=0}^{\infty} c_{n+k} (z-p)^n,$$

where k is the smallest n such that $c_n \neq 0$ (hence the "Furthermore").

Clearly $k > 0$.

The series $\sum_{n=0}^{\infty} c_{n+k}(z-p)^n$ is equal to $\frac{f(z)}{(z-p)^k}$ on $\Delta_r(p) \setminus \{p\}$.

So this series gives g near p. Let $g(p) = c_k$, and $g \colon U \to \mathbb{C}$ is holomorphic.

Uniqueness of k:

Proof of the lemma: On $U \setminus \{p\}$, $g(z) = \frac{f(z)}{(z-p)^k}$ is holomorphic for any k.

For z in some disc $\Delta_r(p)$,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-p)^n = \sum_{n=k}^{\infty} c_n (z-p)^n = (z-p)^k \sum_{n=0}^{\infty} c_{n+k} (z-p)^n,$$

where k is the smallest n such that $c_n \neq 0$ (hence the "Furthermore").

Clearly $k > 0$.

The series $\sum_{n=0}^{\infty} c_{n+k}(z-p)^n$ is equal to $\frac{f(z)}{(z-p)^k}$ on $\Delta_r(p) \setminus \{p\}$.

So this series gives g near p. Let $g(p) = c_k$, and $g\colon U \to \mathbb{C}$ is holomorphic.

Uniqueness of k: Suppose $(z-p)^{k_1} g_1(z) = (z-p)^{k_2} g_2(z)$, where $g_1(p) \neq 0$ and $g_2(p) \neq 0$.

Proof of the lemma: On $U \setminus \{p\}$, $g(z) = \frac{f(z)}{(z-p)^k}$ is holomorphic for any k.

For z in some disc $\Delta_r(p)$,

$$f(z) = \sum_{n=0}^{\infty} c_n(z-p)^n = \sum_{n=k}^{\infty} c_n(z-p)^n = (z-p)^k \sum_{n=0}^{\infty} c_{n+k}(z-p)^n,$$

where k is the smallest n such that $c_n \neq 0$ (hence the "Furthermore").

Clearly $k > 0$.

The series $\sum_{n=0}^{\infty} c_{n+k}(z-p)^n$ is equal to $\frac{f(z)}{(z-p)^k}$ on $\Delta_r(p) \setminus \{p\}$.

So this series gives g near p. Let $g(p) = c_k$, and $g\colon U \to \mathbb{C}$ is holomorphic.

Uniqueness of k: Suppose $(z-p)^{k_1} g_1(z) = (z-p)^{k_2} g_2(z)$, where $g_1(p) \neq 0$ and $g_2(p) \neq 0$.

WLOG suppose $k_1 \leq k_2$,

Proof of the lemma: On $U \setminus \{p\}$, $g(z) = \frac{f(z)}{(z-p)^k}$ is holomorphic for any k.

For z in some disc $\Delta_r(p)$,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-p)^n = \sum_{n=k}^{\infty} c_n (z-p)^n = (z-p)^k \sum_{n=0}^{\infty} c_{n+k} (z-p)^n,$$

where k is the smallest n such that $c_n \neq 0$ (hence the "Furthermore").

Clearly $k > 0$.

The series $\sum_{n=0}^{\infty} c_{n+k} (z-p)^n$ is equal to $\frac{f(z)}{(z-p)^k}$ on $\Delta_r(p) \setminus \{p\}$.

So this series gives g near p. Let $g(p) = c_k$, and $g \colon U \to \mathbb{C}$ is holomorphic.

Uniqueness of k: Suppose $(z-p)^{k_1} g_1(z) = (z-p)^{k_2} g_2(z)$, where $g_1(p) \neq 0$ and $g_2(p) \neq 0$.

WLOG suppose $k_1 \leq k_2$, then $g_1(z) = (z-p)^{k_2-k_1} g_2(z)$.

Proof of the lemma: On $U \setminus \{p\}$, $g(z) = \frac{f(z)}{(z-p)^k}$ is holomorphic for any k.

For z in some disc $\Delta_r(p)$,

$$f(z) = \sum_{n=0}^{\infty} c_n (z-p)^n = \sum_{n=k}^{\infty} c_n (z-p)^n = (z-p)^k \sum_{n=0}^{\infty} c_{n+k} (z-p)^n,$$

where k is the smallest n such that $c_n \neq 0$ (hence the "Furthermore").

Clearly $k > 0$.

The series $\sum_{n=0}^{\infty} c_{n+k}(z-p)^n$ is equal to $\frac{f(z)}{(z-p)^k}$ on $\Delta_r(p) \setminus \{p\}$.

So this series gives g near p. Let $g(p) = c_k$, and $g \colon U \to \mathbb{C}$ is holomorphic.

Uniqueness of k: Suppose $(z-p)^{k_1} g_1(z) = (z-p)^{k_2} g_2(z)$, where $g_1(p) \neq 0$ and $g_2(p) \neq 0$.

WLOG suppose $k_1 \leq k_2$, then $g_1(z) = (z-p)^{k_2-k_1} g_2(z)$.

Plug in $z = p$ to see $k_2 = k_1$. $\square$

Near a zero of order k, a holomorphic function acts like z^k acts near the origin:

Near a zero of order k, a holomorphic function acts like z^k acts near the origin:

Theorem

Suppose $U \subset \mathbb{C}$ is open, $f : U \to \mathbb{C}$ is holomorphic, and $p \in U$ is a zero of f of order $k \in \mathbb{N}$ (an honest zero).

Near a zero of order k, a holomorphic function acts like z^k acts near the origin:

Theorem

Suppose $U \subset \mathbb{C}$ is open, $f : U \to \mathbb{C}$ is holomorphic, and $p \in U$ is a zero of f of order $k \in \mathbb{N}$ (an honest zero). Then there exists an open neighborhood V of p and a holomorphic $g : V \to \mathbb{C}$ such that

$$f(z) = \big(g(z)\big)^k, \qquad \text{where } g(p) = 0 \text{ and } g'(p) \neq 0.$$

Near a zero of order k, a holomorphic function acts like z^k acts near the origin:

Theorem

Suppose $U \subset \mathbb{C}$ is open, $f: U \to \mathbb{C}$ is holomorphic, and $p \in U$ is a zero of f of order $k \in \mathbb{N}$ (an honest zero). Then there exists an open neighborhood V of p and a holomorphic $g: V \to \mathbb{C}$ such that

$$f(z) = \big(g(z)\big)^k, \qquad \text{where } g(p) = 0 \text{ and } g'(p) \neq 0.$$

More fancy: g is a local biholomorphic change of variables near p that makes p into the origin, and makes f into z^k.

Near a zero of order k, a holomorphic function acts like z^k acts near the origin:

Theorem

Suppose $U \subset \mathbb{C}$ is open, $f : U \to \mathbb{C}$ is holomorphic, and $p \in U$ is a zero of f of order $k \in \mathbb{N}$ (an honest zero). Then there exists an open neighborhood V of p and a holomorphic $g : V \to \mathbb{C}$ such that

$$f(z) = \big(g(z)\big)^k, \qquad \text{where } g(p) = 0 \text{ and } g'(p) \neq 0.$$

More fancy: g is a local biholomorphic change of variables near p that makes p into the origin, and makes f into z^k.

Proof: Let $V = \Delta_r(p)$ be such that $f(z) \neq 0$ for any $z \in \Delta_r(p) \setminus \{p\}$.

Near a zero of order k, a holomorphic function acts like z^k acts near the origin:

Theorem

Suppose $U \subset \mathbb{C}$ is open, $f \colon U \to \mathbb{C}$ is holomorphic, and $p \in U$ is a zero of f of order $k \in \mathbb{N}$ (an honest zero). Then there exists an open neighborhood V of p and a holomorphic $g \colon V \to \mathbb{C}$ such that

$$f(z) = \big(g(z)\big)^k, \qquad \text{where } g(p) = 0 \text{ and } g'(p) \neq 0.$$

More fancy: g is a local biholomorphic change of variables near p that makes p into the origin, and makes f into z^k.

Proof: Let $V = \Delta_r(p)$ be such that $f(z) \neq 0$ for any $z \in \Delta_r(p) \setminus \{p\}$.

Lemma says $\exists$ holomorphic $h \colon \Delta_r(p) \to \mathbb{C}$ such that $h(p) \neq 0$ and $f(z) = (z-p)^k h(z)$.

Near a zero of order k, a holomorphic function acts like z^k acts near the origin:

Theorem

Suppose $U \subset \mathbb{C}$ is open, $f: U \to \mathbb{C}$ is holomorphic, and $p \in U$ is a zero of f of order $k \in \mathbb{N}$ (an honest zero). Then there exists an open neighborhood V of p and a holomorphic $g: V \to \mathbb{C}$ such that

$$f(z) = \big(g(z)\big)^k, \qquad \text{where } g(p) = 0 \text{ and } g'(p) \neq 0.$$

More fancy: g is a local biholomorphic change of variables near p that makes p into the origin, and makes f into z^k.

Proof: Let $V = \Delta_r(p)$ be such that $f(z) \neq 0$ for any $z \in \Delta_r(p) \setminus \{p\}$.

Lemma says $\exists$ holomorphic $h: \Delta_r(p) \to \mathbb{C}$ such that $h(p) \neq 0$ and $f(z) = (z-p)^k h(z)$.

In particular, $h(z) \neq 0$ for any $z \in \Delta_r(p)$ (h is nowhere zero).

Near a zero of order k, a holomorphic function acts like z^k acts near the origin:

Theorem

Suppose $U \subset \mathbb{C}$ is open, $f : U \to \mathbb{C}$ is holomorphic, and $p \in U$ is a zero of f of order $k \in \mathbb{N}$ (an honest zero). Then there exists an open neighborhood V of p and a holomorphic $g : V \to \mathbb{C}$ such that

$$f(z) = \big(g(z)\big)^k, \qquad \text{where } g(p) = 0 \text{ and } g'(p) \neq 0.$$

More fancy: g is a local biholomorphic change of variables near p that makes p into the origin, and makes f into z^k.

Proof: Let $V = \Delta_r(p)$ be such that $f(z) \neq 0$ for any $z \in \Delta_r(p) \setminus \{p\}$.

Lemma says $\exists$ holomorphic $h : \Delta_r(p) \to \mathbb{C}$ such that $h(p) \neq 0$ and $f(z) = (z-p)^k h(z)$.

In particular, $h(z) \neq 0$ for any $z \in \Delta_r(p)$ (h is nowhere zero).

As $\Delta_r(p)$ is simply connected $\Rightarrow \exists$ holomorphic $\varphi : \Delta_r(p) \to \mathbb{C}$ such that $\varphi^k = h$.

Near a zero of order k, a holomorphic function acts like z^k acts near the origin:

Theorem

Suppose $U \subset \mathbb{C}$ is open, $f: U \to \mathbb{C}$ is holomorphic, and $p \in U$ is a zero of f of order $k \in \mathbb{N}$ (an honest zero). Then there exists an open neighborhood V of p and a holomorphic $g: V \to \mathbb{C}$ such that

$$f(z) = \big(g(z)\big)^k, \qquad \textit{where } g(p) = 0 \textit{ and } g'(p) \neq 0.$$

More fancy: g is a local biholomorphic change of variables near p that makes p into the origin, and makes f into z^k.

Proof: Let $V = \Delta_r(p)$ be such that $f(z) \neq 0$ for any $z \in \Delta_r(p) \setminus \{p\}$.

Lemma says $\exists$ holomorphic $h: \Delta_r(p) \to \mathbb{C}$ such that $h(p) \neq 0$ and $f(z) = (z-p)^k h(z)$.

In particular, $h(z) \neq 0$ for any $z \in \Delta_r(p)$ (h is nowhere zero).

As $\Delta_r(p)$ is simply connected $\Rightarrow \exists$ holomorphic $\varphi: \Delta_r(p) \to \mathbb{C}$ such that $\varphi^k = h$.

Let $g(z) = (z-p)\varphi(z)$. (So $g^k = f$ and $g(p) = 0$)

Near a zero of order k, a holomorphic function acts like z^k acts near the origin:

Theorem

Suppose $U \subset \mathbb{C}$ is open, $f : U \to \mathbb{C}$ is holomorphic, and $p \in U$ is a zero of f of order $k \in \mathbb{N}$ (an honest zero). Then there exists an open neighborhood V of p and a holomorphic $g : V \to \mathbb{C}$ such that

$$f(z) = \big(g(z)\big)^k, \qquad \text{where } g(p) = 0 \text{ and } g'(p) \neq 0.$$

More fancy: g is a local biholomorphic change of variables near p that makes p into the origin, and makes f into z^k.

Proof: Let $V = \Delta_r(p)$ be such that $f(z) \neq 0$ for any $z \in \Delta_r(p) \setminus \{p\}$.

Lemma says $\exists$ holomorphic $h : \Delta_r(p) \to \mathbb{C}$ such that $h(p) \neq 0$ and $f(z) = (z-p)^k h(z)$.

In particular, $h(z) \neq 0$ for any $z \in \Delta_r(p)$ (h is nowhere zero).

As $\Delta_r(p)$ is simply connected $\Rightarrow \exists$ holomorphic $\varphi : \Delta_r(p) \to \mathbb{C}$ such that $\varphi^k = h$.

Let $g(z) = (z-p)\varphi(z)$.   (So $g^k = f$ and $g(p) = 0$)

As $g'(z) = (z-p)\varphi'(z) + \varphi(z)$, we have $g'(p) = \varphi(p) \neq 0$. □

Every semester, someone wants to use L'Hôpital on homework without proof. You can prove it now.

Every semester, someone wants to use L'Hôpital on homework without proof. You can prove it now.

Exercise: Prove L'Hôpital's rule: If f and g are holomorphic near p, both with an isolated zero at p, and $\lim_{z \to p} \frac{f'(z)}{g'(z)}$ exists (including possibly ∞), then $\lim_{z \to p} \frac{f(z)}{g(z)}$ exists and equals the same thing.