Cultivating Complex Analysis: Rouché's theorem (5.4.2)

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Corollary (Rouché)

Let U, Γ and V be as in the theorem. Suppose $f : U \to \mathbb{C}$ and $g : U \to \mathbb{C}$ are holomorphic such that $|f(z) - g(z)| < |f(z)| + |g(z)|$ for all $z \in \Gamma$. Then f and g have the same number of zeros (counting multiplicity) in V.

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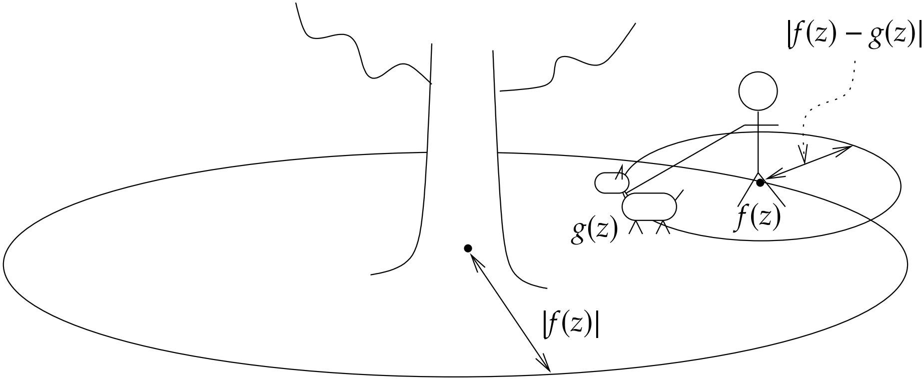
It has a nice geometric interpretation:



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$\square$

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On $\partial\Delta_{1+\epsilon}(0)$,

$$|P(z) - z^n| = 1 < |z^n|\,.$$

By Rouché, $P(z)$ and z^n have the same number of zeros in $\Delta_{1+\epsilon}(0)$.

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If $|z| = 46 + \epsilon$, then

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(Actually the largest zero of P has modulus less than 10).