Cultivating Complex Analysis: Definition (of analytic functions) (2.4.1) Analytic functions are holomorphic (2.4.2)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

Analytic functions are functions equal to a convergent power series near every point.

Definition

Let $U \subset \mathbb{C}$ be open. A function $f\colon U \to \mathbb{C}$ is *analytic* if for every $p \in U$, there exists an $r > 0$ and a power series $\sum c_n (z-p)^n$ converging to f on $\Delta_r(p) \subset U$.

Analytic functions are functions equal to a convergent power series near every point.

Definition

Let $U \subset \mathbb{C}$ be open. A function $f: U \to \mathbb{C}$ is *analytic* if for every $p \in U$, there exists an $r > 0$ and a power series $\sum c_n (z-p)^n$ converging to f on $\Delta_r(p) \subset U$.

Eventually, we will see that function is holomorphic if and only if it is analytic.

Analytic functions are functions equal to a convergent power series near every point.

Definition

Let $U \subset \mathbb{C}$ be open. A function $f : U \to \mathbb{C}$ is *analytic* if for every $p \in U$, there exists an $r > 0$ and a power series $\sum c_n (z-p)^n$ converging to f on $\Delta_r(p) \subset U$.

Eventually, we will see that function is holomorphic if and only if it is analytic.

But that is not so easy. Today we will prove that analytic $\Rightarrow$ holomorphic.

Analytic functions are functions equal to a convergent power series near every point.

Definition

Let $U \subset \mathbb{C}$ be open. A function $f \colon U \to \mathbb{C}$ is *analytic* if for every $p \in U$, there exists an $r > 0$ and a power series $\sum c_n (z-p)^n$ converging to f on $\Delta_r(p) \subset U$.

Eventually, we will see that function is holomorphic if and only if it is analytic.

But that is not so easy. Today we will prove that analytic $\Rightarrow$ holomorphic.

Remark: A subtle point is that it is not immediate that a convergent power series is analytic.

Proposition

Let $f : \Delta_R(p) \to \mathbb{C}$ *be defined by*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-p)^n, \qquad \text{converging in } \Delta_R(p).$$

Proposition

Let $f : \Delta_R(p) \longrightarrow \mathbb{C}$ *be defined by*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-p)^n, \qquad \textit{converging in } \Delta_R(p).$$

Then f *is complex differentiable at every* $z \in \Delta_R(p)$*, and*

$$f'(z) = \sum_{n=1}^{\infty} n c_n (z-p)^{n-1}, \qquad \textit{converging in } \Delta_R(p).$$

Proposition

Let $f : \Delta_R(p) \to \mathbb{C}$ *be defined by*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-p)^n, \qquad \textit{converging in } \Delta_R(p).$$

Then f *is complex differentiable at every* $z \in \Delta_R(p)$*, and*

$$f'(z) = \sum_{n=1}^{\infty} n c_n (z-p)^{n-1}, \qquad \textit{converging in } \Delta_R(p).$$

Proof: WLOG $p = 0$.

Proposition

Let $f : \Delta_R(p) \to \mathbb{C}$ be defined by

$$f(z) = \sum_{n=0}^{\infty} c_n (z-p)^n, \qquad \text{converging in } \Delta_R(p).$$

Then f is complex differentiable at every $z \in \Delta_R(p)$, and

$$f'(z) = \sum_{n=1}^{\infty} n c_n (z-p)^{n-1}, \qquad \text{converging in } \Delta_R(p).$$

Proof: WLOG $p = 0$.

Consider the difference quotient of z^n at z_0

$$\frac{z^n - z_0^n}{z - z_0} = \sum_{k=0}^{n-1} z^k z_0^{n-1-k}$$

Proposition

Let $f : \Delta_R(p) \to \mathbb{C}$ be defined by

$$f(z) = \sum_{n=0}^{\infty} c_n (z-p)^n, \qquad \text{converging in } \Delta_R(p).$$

Then f is complex differentiable at every $z \in \Delta_R(p)$, and

$$f'(z) = \sum_{n=1}^{\infty} n c_n (z-p)^{n-1}, \qquad \text{converging in } \Delta_R(p).$$

Proof: WLOG $p = 0$.

Consider the difference quotient of z^n at z_0

$$\frac{z^n - z_0^n}{z - z_0} = \sum_{k=0}^{n-1} z^k z_0^{n-1-k} \quad \underset{\text{as } z \to z_0}{\longrightarrow} \quad n z_0^{n-1}.$$

Suppose $z_0, z \in \Delta_R(0)$,

$$\frac{f(z) - f(z_0)}{z - z_0}$$

Suppose $z_0, z \in \Delta_R(0)$,

$$\frac{f(z) - f(z_0)}{z - z_0} = \sum_{n=1}^{\infty} c_n \frac{z^n - z_0^n}{z - z_0}$$

Suppose $z_0, z \in \Delta_R(0)$,

$$\frac{f(z) - f(z_0)}{z - z_0} = \sum_{n=1}^{\infty} c_n \frac{z^n - z_0^n}{z - z_0} = \sum_{n=1}^{\infty} c_n \sum_{k=0}^{n-1} z^k z_0^{n-1-k}.$$

Suppose $z_0, z \in \Delta_R(0)$,

$$\frac{f(z) - f(z_0)}{z - z_0} = \sum_{n=1}^{\infty} c_n \frac{z^n - z_0^n}{z - z_0} = \sum_{n=1}^{\infty} c_n \sum_{k=0}^{n-1} z^k z_0^{n-1-k}.$$

We need to show that the expression on the right is a continuous function of z at z_0.

Suppose $z_0, z \in \Delta_R(0)$,

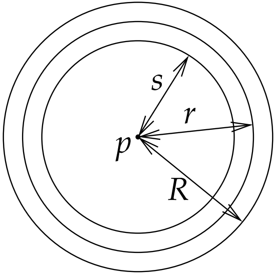
$$\frac{f(z) - f(z_0)}{z - z_0} = \sum_{n=1}^{\infty} c_n \frac{z^n - z_0^n}{z - z_0} = \sum_{n=1}^{\infty} c_n \sum_{k=0}^{n-1} z^k z_0^{n-1-k}.$$

We need to show that the expression on the right is a continuous function of z at z_0. It is continuous if the series converges uniformly for z in a neighborhood of z_0.

Suppose $z_0, z \in \Delta_R(0)$,

$$\frac{f(z) - f(z_0)}{z - z_0} = \sum_{n=1}^{\infty} c_n \frac{z^n - z_0^n}{z - z_0} = \sum_{n=1}^{\infty} c_n \sum_{k=0}^{n-1} z^k z_0^{n-1-k}.$$

We need to show that the expression on the right is a continuous function of z at z_0.
It is continuous if the series converges uniformly for z in a neighborhood of z_0.
Suppose $0 < s < r < R$ and $z_0, z \in \Delta_s(0)$.

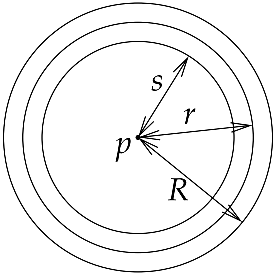


Suppose $z_0, z \in \Delta_R(0)$,

$$\frac{f(z) - f(z_0)}{z - z_0} = \sum_{n=1}^{\infty} c_n \frac{z^n - z_0^n}{z - z_0} = \sum_{n=1}^{\infty} c_n \sum_{k=0}^{n-1} z^k z_0^{n-1-k}.$$

We need to show that the expression on the right is a continuous function of z at z_0.
It is continuous if the series converges uniformly for z in a neighborhood of z_0.
Suppose $0 < s < r < R$ and $z_0, z \in \Delta_s(0)$.

$$\left| c_n \sum_{k=0}^{n-1} z^k z_0^{n-1-k} \right| \leq \sum_{k=0}^{n-1} |c_n| s^{n-1} = n|c_n|s^{n-1} = n|c_n|r^{n-1}\left(\frac{s}{r}\right)^{n-1}$$

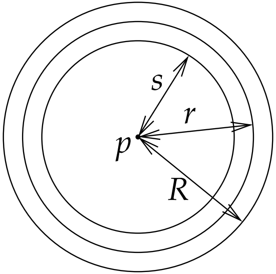


Suppose $z_0, z \in \Delta_R(0)$,

$$\frac{f(z) - f(z_0)}{z - z_0} = \sum_{n=1}^{\infty} c_n \frac{z^n - z_0^n}{z - z_0} = \sum_{n=1}^{\infty} c_n \sum_{k=0}^{n-1} z^k z_0^{n-1-k}.$$

We need to show that the expression on the right is a continuous function of z at z_0.
It is continuous if the series converges uniformly for z in a neighborhood of z_0.
Suppose $0 < s < r < R$ and $z_0, z \in \Delta_s(0)$.

$$\left| c_n \sum_{k=0}^{n-1} z^k z_0^{n-1-k} \right| \leq \sum_{k=0}^{n-1} |c_n| s^{n-1} = n|c_n|s^{n-1} = n|c_n|r^{n-1}\left(\frac{s}{r}\right)^{n-1}$$

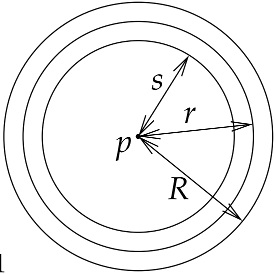


$|c_n|r^{n-1}$ is bounded by some $M > 0$ (f converges in $\Delta_R(0)$ and $r < R$)

Suppose $z_0, z \in \Delta_R(0)$,

$$\frac{f(z) - f(z_0)}{z - z_0} = \sum_{n=1}^{\infty} c_n \frac{z^n - z_0^n}{z - z_0} = \sum_{n=1}^{\infty} c_n \sum_{k=0}^{n-1} z^k z_0^{n-1-k}.$$

We need to show that the expression on the right is a continuous function of z at z_0.
It is continuous if the series converges uniformly for z in a neighborhood of z_0.
Suppose $0 < s < r < R$ and $z_0, z \in \Delta_s(0)$.



$$\left| c_n \sum_{k=0}^{n-1} z^k z_0^{n-1-k} \right| \leq \sum_{k=0}^{n-1} |c_n| s^{n-1} = n|c_n| s^{n-1} = n|c_n| r^{n-1} \left(\frac{s}{r}\right)^{n-1}$$

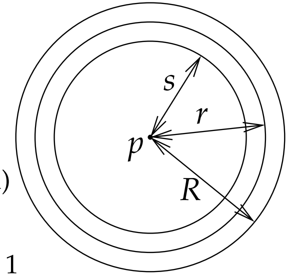
$|c_n| r^{n-1}$ is bounded by some $M > 0$ (f converges in $\Delta_R(0)$ and $r < R$)

$$\sqrt[n]{n|c_n|s^{n-1}} = \sqrt[n]{n|c_n|r^{n-1}\left(\frac{s}{r}\right)^{n-1}} \leq \sqrt[n]{nM\left(\frac{s}{r}\right)^{n-1}} \quad \underset{\text{as } n \to \infty}{\longrightarrow} \quad \frac{s}{r} < 1$$

Suppose $z_0, z \in \Delta_R(0)$,

$$\frac{f(z) - f(z_0)}{z - z_0} = \sum_{n=1}^{\infty} c_n \frac{z^n - z_0^n}{z - z_0} = \sum_{n=1}^{\infty} c_n \sum_{k=0}^{n-1} z^k z_0^{n-1-k}.$$

We need to show that the expression on the right is a continuous function of z at z_0.
It is continuous if the series converges uniformly for z in a neighborhood of z_0.
Suppose $0 < s < r < R$ and $z_0, z \in \Delta_s(0)$.



$$\left| c_n \sum_{k=0}^{n-1} z^k z_0^{n-1-k} \right| \le \sum_{k=0}^{n-1} |c_n| s^{n-1} = n|c_n| s^{n-1} = n|c_n| r^{n-1} \left(\frac{s}{r}\right)^{n-1}$$

$|c_n| r^{n-1}$ is bounded by some $M > 0$ (f converges in $\Delta_R(0)$ and $r < R$)

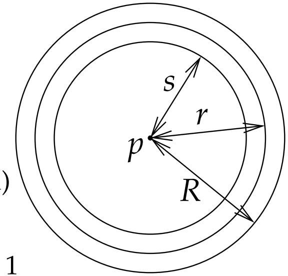
$$\sqrt[n]{n|c_n|s^{n-1}} = \sqrt[n]{n|c_n|r^{n-1}\left(\frac{s}{r}\right)^{n-1}} \le \sqrt[n]{nM\left(\frac{s}{r}\right)^{n-1}} \underset{\text{as } n\to\infty}{\longrightarrow} \frac{s}{r} < 1$$

so $\sum n|c_n|s^{n-1}$ converges (root test).

Suppose $z_0, z \in \Delta_R(0)$,

$$\frac{f(z) - f(z_0)}{z - z_0} = \sum_{n=1}^{\infty} c_n \frac{z^n - z_0^n}{z - z_0} = \sum_{n=1}^{\infty} c_n \sum_{k=0}^{n-1} z^k z_0^{n-1-k}.$$

We need to show that the expression on the right is a continuous function of z at z_0.
It is continuous if the series converges uniformly for z in a neighborhood of z_0.
Suppose $0 < s < r < R$ and $z_0, z \in \Delta_s(0)$.



$$\left| c_n \sum_{k=0}^{n-1} z^k z_0^{n-1-k} \right| \leq \sum_{k=0}^{n-1} |c_n| s^{n-1} = n|c_n|s^{n-1} = n|c_n|r^{n-1}\left(\frac{s}{r}\right)^{n-1}$$

$|c_n|r^{n-1}$ is bounded by some $M > 0$ (f converges in $\Delta_R(0)$ and $r < R$)

$$\sqrt[n]{n|c_n|s^{n-1}} = \sqrt[n]{n|c_n|r^{n-1}\left(\frac{s}{r}\right)^{n-1}} \leq \sqrt[n]{nM\left(\frac{s}{r}\right)^{n-1}} \quad \underset{\text{as } n \to \infty}{\longrightarrow} \quad \frac{s}{r} < 1$$

so $\sum n|c_n|s^{n-1}$ converges (root test).
So $\sum_{n=1}^{\infty} c_n \sum_{k=0}^{n-1} z^k z_0^{n-1-k}$, converges uniformly in z on $\Delta_s(p)$.

So we can swap the limits:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

So we can swap the limits:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \sum_{n=1}^{\infty} c_n \sum_{k=0}^{n-1} z_0^k z_0^{n-1-k} = \sum_{n=1}^{\infty} n c_n z_0^{n-1}.$$

So we can swap the limits:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \sum_{n=1}^{\infty} c_n \sum_{k=0}^{n-1} z_0^k z_0^{n-1-k} = \sum_{n=1}^{\infty} n c_n z_0^{n-1}.$$

As s and r were arbitrary we get the result at any $z_0 \in \Delta_R(0)$. □

So we can swap the limits:

$$\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \sum_{n=1}^{\infty} c_n \sum_{k=0}^{n-1} z_0^k z_0^{n-1-k} = \sum_{n=1}^{\infty} n c_n z_0^{n-1}.$$

As s and r were arbitrary we get the result at any $z_0 \in \Delta_R(0)$. $\square$

Corollary

Let $f : \Delta_R(p) \to \mathbb{C}$ *be defined by* $f(z) = \sum_{n=0}^{\infty} c_n (z-p)^n$*, converging in* $\Delta_R(p)$.

So we can swap the limits:

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\sum_{n=1}^{\infty}c_n\sum_{k=0}^{n-1}z_0^k z_0^{n-1-k}=\sum_{n=1}^{\infty}nc_n z_0^{n-1}.$$

As s and r were arbitrary we get the result at any $z_0\in\Delta_R(0)$. □

Corollary

Let $f:\Delta_R(p)\to\mathbb{C}$ *be defined by* $f(z)=\sum_{n=0}^{\infty}c_n(z-p)^n$*, converging in* $\Delta_R(p)$*.*

Then f *is infinitely complex differentiable in* $\Delta_R(p)$*, and the* k^{th} *derivative is given by*

$$f^{(k)}(z)=\sum_{n=k}^{\infty}n(n-1)\cdots(n-k+1)c_n(z-p)^{n-k},\qquad \textit{converging in } \Delta_R(p).$$

So we can swap the limits:

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \sum_{n=1}^{\infty} c_n \sum_{k=0}^{n-1} z_0^k z_0^{n-1-k} = \sum_{n=1}^{\infty} n c_n z_0^{n-1}.$$

As s and r were arbitrary we get the result at any $z_0 \in \Delta_R(0)$. □

Corollary

Let $f : \Delta_R(p) \to \mathbb{C}$ be defined by $f(z) = \sum_{n=0}^{\infty} c_n (z-p)^n$, converging in $\Delta_R(p)$.

Then f is infinitely complex differentiable in $\Delta_R(p)$, and the k^{th} derivative is given by

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) c_n (z-p)^{n-k}, \qquad \text{converging in } \Delta_R(p).$$

Furthermore, $\quad c_n = \dfrac{f^{(n)}(p)}{n!}.$

A consequence is that the power series is unique since the coefficients are unique

$$c_n = \frac{f^{(n)}(p)}{n!}.$$

A consequence is that the power series is unique since the coefficients are unique

$$c_n = \frac{f^{(n)}(p)}{n!}.$$

In fact, c_n depends only on values of f in arbitrarily small neighborhood of p.

A consequence is that the power series is unique since the coefficients are unique

$$c_n = \frac{f^{(n)}(p)}{n!}.$$

In fact, c_n depends only on values of f in arbitrarily small neighborhood of p.

Applied to the analytic functions we get:

Corollary

An analytic function is infinitely complex differentiable, and each derivative is analytic.