Cultivating Complex Analysis: Simply connected domains (4.3 part 1)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

Definition

A domain $U \subset \mathbb{C}$ is *simply connected* if every cycle in U is homologous to zero in U.

Definition

A domain $U \subset \mathbb{C}$ is *simply connected* if every cycle in U is homologous to zero in U.

That is, U is simply connected if $n(\Gamma; p) = 0$ for every cycle Γ in U and every $p \in \mathbb{C} \setminus U$.

Definition

A domain $U \subset \mathbb{C}$ is *simply connected* if every cycle in U is homologous to zero in U.

That is, U is simply connected if $n(\Gamma; p) = 0$ for every cycle Γ in U and every $p \in \mathbb{C} \setminus U$.

Exercise:
a) star-like domains (e.g., $\mathbb{C}$, $\mathbb{D}$, and $\mathbb{H}$) in $\mathbb{C}$ are simply connected.

Definition

A domain $U \subset \mathbb{C}$ is *simply connected* if every cycle in U is homologous to zero in U.

That is, U is simply connected if $n(\Gamma; p) = 0$ for every cycle Γ in U and every $p \in \mathbb{C} \setminus U$.

Exercise:
a) star-like domains (e.g., $\mathbb{C}$, $\mathbb{D}$, and $\mathbb{H}$) in $\mathbb{C}$ are simply connected.
b) $\mathbb{C} \setminus \{0\}$ is not simply connected.

Definition

A domain $U \subset \mathbb{C}$ is *simply connected* if every cycle in U is homologous to zero in U.

That is, U is simply connected if $n(\Gamma; p) = 0$ for every cycle Γ in U and every $p \in \mathbb{C} \setminus U$.

Exercise:
a) star-like domains (e.g., $\mathbb{C}$, $\mathbb{D}$, and $\mathbb{H}$) in $\mathbb{C}$ are simply connected.
b) $\mathbb{C} \setminus \{0\}$ is not simply connected.

A few remarks are in order:

Definition

A domain $U \subset \mathbb{C}$ is *simply connected* if every cycle in U is homologous to zero in U.

That is, U is simply connected if $n(\Gamma; p) = 0$ for every cycle Γ in U and every $p \in \mathbb{C} \setminus U$.

Exercise:
a) star-like domains (e.g., $\mathbb{C}$, $\mathbb{D}$, and $\mathbb{H}$) in $\mathbb{C}$ are simply connected.
b) $\mathbb{C} \setminus \{0\}$ is not simply connected.

A few remarks are in order:

Remark: This is the "wrong" definition. It happens to work in the setting of domains in $\mathbb{C}$. It is wrong for two reasons:

Definition

A domain $U \subset \mathbb{C}$ is *simply connected* if every cycle in U is homologous to zero in U.

That is, U is simply connected if $n(\Gamma; p) = 0$ for every cycle Γ in U and every $p \in \mathbb{C} \setminus U$.

Exercise:
a) star-like domains (e.g., $\mathbb{C}$, $\mathbb{D}$, and $\mathbb{H}$) in $\mathbb{C}$ are simply connected.
b) $\mathbb{C} \setminus \{0\}$ is not simply connected.

A few remarks are in order:

Remark: This is the "wrong" definition. It happens to work in the setting of domains in $\mathbb{C}$. It is wrong for two reasons:

1) It is in terms of homology and not homotopy (an optional section in the book).

Definition

A domain $U \subset \mathbb{C}$ is *simply connected* if every cycle in U is homologous to zero in U.

That is, U is simply connected if $n(\Gamma; p) = 0$ for every cycle Γ in U and every $p \in \mathbb{C} \setminus U$.

Exercise:
a) star-like domains (e.g., $\mathbb{C}$, $\mathbb{D}$, and $\mathbb{H}$) in $\mathbb{C}$ are simply connected.
b) $\mathbb{C} \setminus \{0\}$ is not simply connected.

A few remarks are in order:

Remark: This is the “wrong” definition. It happens to work in the setting of domains in $\mathbb{C}$. It is wrong for two reasons:

1) It is in terms of homology and not homotopy (an optional section in the book).

One could say *simply connected in the sense of homology* to emphasize.

Definition

A domain $U \subset \mathbb{C}$ is *simply connected* if every cycle in U is homologous to zero in U.

That is, U is simply connected if $n(\Gamma; p) = 0$ for every cycle Γ in U and every $p \in \mathbb{C} \setminus U$.

Exercise:
a) star-like domains (e.g., $\mathbb{C}$, $\mathbb{D}$, and $\mathbb{H}$) in $\mathbb{C}$ are simply connected.
b) $\mathbb{C} \setminus \{0\}$ is not simply connected.

A few remarks are in order:

Remark: This is the "wrong" definition. It happens to work in the setting of domains in $\mathbb{C}$. It is wrong for two reasons:

1) It is in terms of homology and not homotopy (an optional section in the book).

One could say *simply connected in the sense of homology* to emphasize.

2) We defined cycles as "piecewise-C^1" instead of "continuous."

Definition

A domain $U \subset \mathbb{C}$ is *simply connected* if every cycle in U is homologous to zero in U.

That is, U is simply connected if $n(\Gamma; p) = 0$ for every cycle Γ in U and every $p \in \mathbb{C} \setminus U$.

Exercise:
a) star-like domains (e.g., $\mathbb{C}$, $\mathbb{D}$, and $\mathbb{H}$) in $\mathbb{C}$ are simply connected.
b) $\mathbb{C} \setminus \{0\}$ is not simply connected.

A few remarks are in order:

Remark: This is the "wrong" definition. It happens to work in the setting of domains in $\mathbb{C}$. It is wrong for two reasons:

1) It is in terms of homology and not homotopy (an optional section in the book).

One could say *simply connected in the sense of homology* to emphasize.

2) We defined cycles as "piecewise-C^1" instead of "continuous."

Remark: Can a disconnected set be simply connected? We remain neutral on this.

Simply connected domains satisfy Cauchy's theorem.

Theorem (Cauchy's theorem (simply connected version))

Let $U \subset \mathbb{C}$ *be a simply connected domain and* $f : U \to \mathbb{C}$ *holomorphic.*

Simply connected domains satisfy Cauchy's theorem.

Theorem (Cauchy's theorem (simply connected version))

Let $U \subset \mathbb{C}$ be a simply connected domain and $f : U \to \mathbb{C}$ holomorphic. If Γ is a cycle in U, then

$$\int_\Gamma f(z)\,dz = 0.$$

Simply connected domains satisfy Cauchy's theorem.

Theorem (Cauchy's theorem (simply connected version))

Let $U \subset \mathbb{C}$ be a simply connected domain and $f : U \to \mathbb{C}$ holomorphic. If Γ is a cycle in U, then

$$\int_\Gamma f(z)\,dz = 0.$$

Proof: Since U is simply connected, $n(\Gamma; p) = 0$ for every $p \in \mathbb{C} \setminus U$, so Cauchy applies. □

Simply connected domains satisfy Cauchy's theorem.

Theorem (Cauchy's theorem (simply connected version))

Let $U \subset \mathbb{C}$ be a simply connected domain and $f : U \to \mathbb{C}$ holomorphic. If Γ is a cycle in U, then

$$\int_\Gamma f(z)\,dz = 0.$$

Proof: Since U is simply connected, $n(\Gamma; p) = 0$ for every $p \in \mathbb{C} \setminus U$, so Cauchy applies. □

If we have Cauchy's theorem we expect primitives:

Theorem

Let $U \subset \mathbb{C}$ be a simply connected domain and $f : U \to \mathbb{C}$ holomorphic.

Simply connected domains satisfy Cauchy's theorem.

Theorem (Cauchy's theorem (simply connected version))

Let $U \subset \mathbb{C}$ be a simply connected domain and $f : U \to \mathbb{C}$ holomorphic. If Γ is a cycle in U, then

$$\int_{\Gamma} f(z)\, dz = 0.$$

Proof: Since U is simply connected, $n(\Gamma; p) = 0$ for every $p \in \mathbb{C} \setminus U$, so Cauchy applies. $\square$

If we have Cauchy's theorem we expect primitives:

Theorem

Let $U \subset \mathbb{C}$ be a simply connected domain and $f : U \to \mathbb{C}$ holomorphic. Then f has a primitive in U.

Simply connected domains satisfy Cauchy's theorem.

Theorem (Cauchy's theorem (simply connected version))

Let $U \subset \mathbb{C}$ be a simply connected domain and $f : U \to \mathbb{C}$ holomorphic. If Γ is a cycle in U, then

$$\int_\Gamma f(z)\,dz = 0.$$

Proof: Since U is simply connected, $n(\Gamma; p) = 0$ for every $p \in \mathbb{C} \setminus U$, so Cauchy applies. □

If we have Cauchy's theorem we expect primitives:

Theorem

Let $U \subset \mathbb{C}$ be a simply connected domain and $f : U \to \mathbb{C}$ holomorphic. Then f has a primitive in U.

Proof: Fix $p \in U$ and note that U is path connected.

Simply connected domains satisfy Cauchy's theorem.

Theorem (Cauchy's theorem (simply connected version))

Let $U \subset \mathbb{C}$ be a simply connected domain and $f : U \to \mathbb{C}$ holomorphic. If Γ is a cycle in U, then

$$\int_\Gamma f(z)\,dz = 0.$$

Proof: Since U is simply connected, $n(\Gamma; p) = 0$ for every $p \in \mathbb{C} \setminus U$, so Cauchy applies. □

If we have Cauchy's theorem we expect primitives:

Theorem

Let $U \subset \mathbb{C}$ be a simply connected domain and $f : U \to \mathbb{C}$ holomorphic. Then f has a primitive in U.

Proof: Fix $p \in U$ and note that U is path connected.
For every $z \in U$, pick some piecewise-C^1 path γ from p to z, and

Simply connected domains satisfy Cauchy's theorem.

Theorem (Cauchy's theorem (simply connected version))

Let $U \subset \mathbb{C}$ be a simply connected domain and $f : U \to \mathbb{C}$ holomorphic. If Γ is a cycle in U, then

$$\int_\Gamma f(z)\, dz = 0.$$

Proof: Since U is simply connected, $n(\Gamma; p) = 0$ for every $p \in \mathbb{C} \setminus U$, so Cauchy applies. □

If we have Cauchy's theorem we expect primitives:

Theorem

Let $U \subset \mathbb{C}$ be a simply connected domain and $f : U \to \mathbb{C}$ holomorphic. Then f has a primitive in U.

Proof: Fix $p \in U$ and note that U is path connected.
For every $z \in U$, pick some piecewise-C^1 path γ from p to z, and

Define $$F(z) = \int_\gamma f(\zeta)\, d\zeta.$$

If α is another path from p to z, then by Cauchy

$$\int_\gamma f(\zeta)\,d\zeta - \int_\alpha f(\zeta)\,d\zeta = \int_{\gamma-\alpha} f(\zeta)\,d\zeta = 0$$

If α is another path from p to z, then by Cauchy

$$\int_\gamma f(\zeta)\,d\zeta - \int_\alpha f(\zeta)\,d\zeta = \int_{\gamma-\alpha} f(\zeta)\,d\zeta = 0 \quad \Rightarrow \quad F(z) = \int_\gamma f(\zeta)\,d\zeta \text{ does not depend on } \gamma.$$

If α is another path from p to z, then by Cauchy

$$\int_\gamma f(\zeta)\, d\zeta - \int_\alpha f(\zeta)\, d\zeta = \int_{\gamma-\alpha} f(\zeta)\, d\zeta = 0 \quad \Rightarrow \quad F(z) = \int_\gamma f(\zeta)\, d\zeta \text{ does not depend on } \gamma.$$

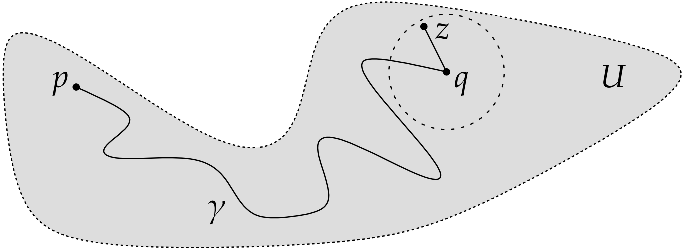
Consider $q \in U$, γ a path from p to q, and $\Delta_r(q) \subset U$.

If α is another path from p to z, then by Cauchy

$$\int_\gamma f(\zeta)\,d\zeta - \int_\alpha f(\zeta)\,d\zeta = \int_{\gamma-\alpha} f(\zeta)\,d\zeta = 0 \quad \Rightarrow \quad F(z) = \int_\gamma f(\zeta)\,d\zeta \text{ does not depend on } \gamma.$$

Consider $q \in U$, γ a path from p to q, and $\Delta_r(q) \subset U$. For $z \in \Delta_r(q)$

$$F(z) = \int_{\gamma+[q,z]} f(\zeta)\,d\zeta$$

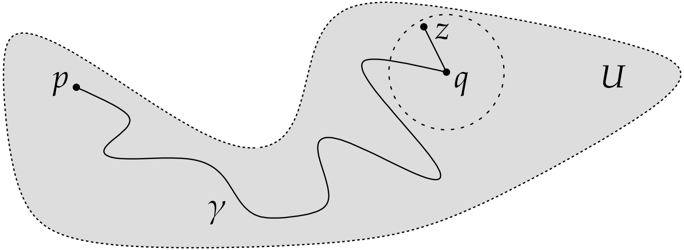


If α is another path from p to z, then by Cauchy

$$\int_\gamma f(\zeta)\,d\zeta - \int_\alpha f(\zeta)\,d\zeta = \int_{\gamma-\alpha} f(\zeta)\,d\zeta = 0 \quad \Rightarrow \quad F(z) = \int_\gamma f(\zeta)\,d\zeta \text{ does not depend on } \gamma.$$

Consider $q \in U$, γ a path from p to q, and $\Delta_r(q) \subset U$. For $z \in \Delta_r(q)$

$$\begin{aligned} F(z) &= \int_{\gamma+[q,z]} f(\zeta)\,d\zeta \\ &= \int_\gamma f(\zeta)\,d\zeta + \int_{[q,z]} f(\zeta)\,d\zeta. \end{aligned}$$

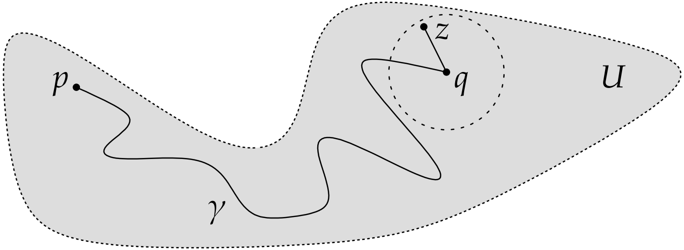


If α is another path from p to z, then by Cauchy

$$\int_\gamma f(\zeta)\,d\zeta - \int_\alpha f(\zeta)\,d\zeta = \int_{\gamma-\alpha} f(\zeta)\,d\zeta = 0 \quad \Rightarrow \quad F(z) = \int_\gamma f(\zeta)\,d\zeta \text{ does not depend on } \gamma.$$

Consider $q \in U$, γ a path from p to q, and $\Delta_r(q) \subset U$. For $z \in \Delta_r(q)$

$$\begin{aligned} F(z) &= \int_{\gamma+[q,z]} f(\zeta)\,d\zeta \\ &= \int_\gamma f(\zeta)\,d\zeta + \int_{[q,z]} f(\zeta)\,d\zeta. \end{aligned}$$



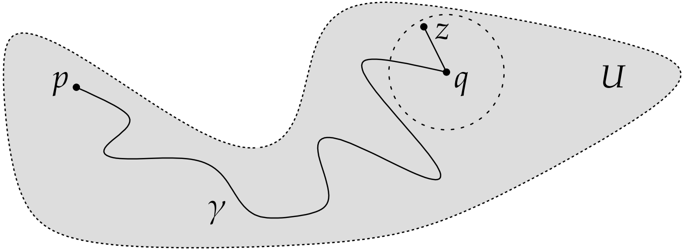
The first term is a constant.

If α is another path from p to z, then by Cauchy

$$\int_{\gamma} f(\zeta)\, d\zeta - \int_{\alpha} f(\zeta)\, d\zeta = \int_{\gamma-\alpha} f(\zeta)\, d\zeta = 0 \quad \Rightarrow \quad F(z) = \int_{\gamma} f(\zeta)\, d\zeta \text{ does not depend on } \gamma.$$

Consider $q \in U$, γ a path from p to q, and $\Delta_r(q) \subset U$. For $z \in \Delta_r(q)$

$$\begin{aligned} F(z) &= \int_{\gamma+[q,z]} f(\zeta)\, d\zeta \\ &= \int_{\gamma} f(\zeta)\, d\zeta + \int_{[q,z]} f(\zeta)\, d\zeta. \end{aligned}$$



The first term is a constant.

The second term is how we defined a primitive in a star-like domain ($\Delta_r(q)$). See Proposition 3.2.11.

□

Corollary

Let $U \subset \mathbb{C}$ *be a simply connected domain and* $f : U \to \mathbb{C}$ *nowhere zero and holomorphic.*

Corollary

Let $U \subset \mathbb{C}$ be a simply connected domain and $f: U \longrightarrow \mathbb{C}$ nowhere zero and holomorphic. Then there exists a holomorphic $g: U \longrightarrow \mathbb{C}$ such that

$$e^{g(z)} = f(z).$$

Corollary

Let $U \subset \mathbb{C}$ be a simply connected domain and $f\colon U \longrightarrow \mathbb{C}$ nowhere zero and holomorphic. Then there exists a holomorphic $g\colon U \longrightarrow \mathbb{C}$ such that

$$e^{g(z)} = f(z).$$

Example: If $U \subset \mathbb{C} \setminus \{0\}$ is a simply connected domain, then $\exists$ a holomorphic $L\colon U \longrightarrow \mathbb{C}$ such that $e^{L(z)} = z$ (a branch of the log).

Corollary

Let $U \subset \mathbb{C}$ be a simply connected domain and $f\colon U \longrightarrow \mathbb{C}$ nowhere zero and holomorphic. Then there exists a holomorphic $g\colon U \longrightarrow \mathbb{C}$ such that

$$e^{g(z)} = f(z).$$

Example: If $U \subset \mathbb{C} \setminus \{0\}$ is a simply connected domain, then $\exists$ a holomorphic $L\colon U \longrightarrow \mathbb{C}$ such that $e^{L(z)} = z$ (a branch of the log).

Proof: $\dfrac{f'(z)}{f(z)}$ is holomorphic on U.

Corollary

Let $U \subset \mathbb{C}$ be a simply connected domain and $f\colon U \longrightarrow \mathbb{C}$ nowhere zero and holomorphic. Then there exists a holomorphic $g\colon U \longrightarrow \mathbb{C}$ such that

$$e^{g(z)} = f(z).$$

Example: If $U \subset \mathbb{C} \setminus \{0\}$ is a simply connected domain, then $\exists$ a holomorphic $L\colon U \longrightarrow \mathbb{C}$ such that $e^{L(z)} = z$ (a branch of the log).

Proof: $\dfrac{f'(z)}{f(z)}$ is holomorphic on U. Find a primitive $g(z)$.

Corollary

Let $U \subset \mathbb{C}$ be a simply connected domain and $f\colon U \longrightarrow \mathbb{C}$ nowhere zero and holomorphic. Then there exists a holomorphic $g\colon U \longrightarrow \mathbb{C}$ such that

$$e^{g(z)} = f(z).$$

Example: If $U \subset \mathbb{C} \setminus \{0\}$ is a simply connected domain, then $\exists$ a holomorphic $L\colon U \longrightarrow \mathbb{C}$ such that $e^{L(z)} = z$ (a branch of the log).

Proof: $\dfrac{f'(z)}{f(z)}$ is holomorphic on U. Find a primitive $g(z)$. Then

$$\frac{d}{dz}\left[\frac{e^{g(z)}}{f(z)}\right]$$

Corollary

Let $U \subset \mathbb{C}$ be a simply connected domain and $f\colon U \longrightarrow \mathbb{C}$ nowhere zero and holomorphic. Then there exists a holomorphic $g\colon U \longrightarrow \mathbb{C}$ such that

$$e^{g(z)} = f(z).$$

Example: If $U \subset \mathbb{C} \setminus \{0\}$ is a simply connected domain, then $\exists$ a holomorphic $L\colon U \longrightarrow \mathbb{C}$ such that $e^{L(z)} = z$ (a branch of the log).

Proof: $\dfrac{f'(z)}{f(z)}$ is holomorphic on U. Find a primitive $g(z)$. Then

$$\frac{d}{dz}\left[\frac{e^{g(z)}}{f(z)}\right] = \frac{e^{g(z)}g'(z)f(z) - e^{g(z)}f'(z)}{\big(f(z)\big)^2}$$

Corollary

Let $U \subset \mathbb{C}$ be a simply connected domain and $f\colon U \to \mathbb{C}$ nowhere zero and holomorphic. Then there exists a holomorphic $g\colon U \to \mathbb{C}$ such that

$$e^{g(z)} = f(z).$$

Example: If $U \subset \mathbb{C} \setminus \{0\}$ is a simply connected domain, then $\exists$ a holomorphic $L\colon U \to \mathbb{C}$ such that $e^{L(z)} = z$ (a branch of the log).

Proof: $\dfrac{f'(z)}{f(z)}$ is holomorphic on U. Find a primitive $g(z)$. Then

$$\frac{d}{dz}\left[\frac{e^{g(z)}}{f(z)}\right] = \frac{e^{g(z)}g'(z)f(z) - e^{g(z)}f'(z)}{\big(f(z)\big)^2} = \frac{e^{g(z)}f'(z) - e^{g(z)}f'(z)}{\big(f(z)\big)^2}$$

Corollary

Let $U \subset \mathbb{C}$ be a simply connected domain and $f\colon U \to \mathbb{C}$ nowhere zero and holomorphic. Then there exists a holomorphic $g\colon U \to \mathbb{C}$ such that

$$e^{g(z)} = f(z).$$

Example: If $U \subset \mathbb{C} \setminus \{0\}$ is a simply connected domain, then $\exists$ a holomorphic $L\colon U \to \mathbb{C}$ such that $e^{L(z)} = z$ (a branch of the log).

Proof: $\dfrac{f'(z)}{f(z)}$ is holomorphic on U. Find a primitive $g(z)$. Then

$$\frac{d}{dz}\left[\frac{e^{g(z)}}{f(z)}\right] = \frac{e^{g(z)}g'(z)f(z) - e^{g(z)}f'(z)}{\big(f(z)\big)^2} = \frac{e^{g(z)}f'(z) - e^{g(z)}f'(z)}{\big(f(z)\big)^2} = 0.$$

Corollary

Let $U \subset \mathbb{C}$ be a simply connected domain and $f\colon U \to \mathbb{C}$ nowhere zero and holomorphic. Then there exists a holomorphic $g\colon U \to \mathbb{C}$ such that

$$e^{g(z)} = f(z).$$

Example: If $U \subset \mathbb{C} \setminus \{0\}$ is a simply connected domain, then $\exists$ a holomorphic $L\colon U \to \mathbb{C}$ such that $e^{L(z)} = z$ (a branch of the log).

Proof: $\dfrac{f'(z)}{f(z)}$ is holomorphic on U. Find a primitive $g(z)$. Then

$$\frac{d}{dz}\left[\frac{e^{g(z)}}{f(z)}\right] = \frac{e^{g(z)}g'(z)f(z) - e^{g(z)}f'(z)}{\big(f(z)\big)^2} = \frac{e^{g(z)}f'(z) - e^{g(z)}f'(z)}{\big(f(z)\big)^2} = 0.$$

$\Rightarrow \quad \dfrac{e^{g(z)}}{f(z)}$ is constant.

Corollary

Let $U \subset \mathbb{C}$ be a simply connected domain and $f\colon U \longrightarrow \mathbb{C}$ nowhere zero and holomorphic. Then there exists a holomorphic $g\colon U \longrightarrow \mathbb{C}$ such that

$$e^{g(z)} = f(z).$$

Example: If $U \subset \mathbb{C} \setminus \{0\}$ is a simply connected domain, then $\exists$ a holomorphic $L\colon U \longrightarrow \mathbb{C}$ such that $e^{L(z)} = z$ (a branch of the log).

Proof: $\dfrac{f'(z)}{f(z)}$ is holomorphic on U. Find a primitive $g(z)$. Then

$$\frac{d}{dz}\left[\frac{e^{g(z)}}{f(z)}\right] = \frac{e^{g(z)}g'(z)f(z) - e^{g(z)}f'(z)}{\big(f(z)\big)^2} = \frac{e^{g(z)}f'(z) - e^{g(z)}f'(z)}{\big(f(z)\big)^2} = 0.$$

$\Rightarrow \quad \dfrac{e^{g(z)}}{f(z)}$ is constant.

$\Rightarrow \quad \exists C \in \mathbb{C}$ such that $e^{g(z)+C} = f(z)$. $\square$

If we have the logarithm, we can take roots.

If we have the logarithm, we can take roots.

Corollary

Let $U \subset \mathbb{C}$ be a simply connected domain, $f : U \to \mathbb{C}$ nowhere zero and holomorphic, and $k \in \mathbb{N}$.

If we have the logarithm, we can take roots.

Corollary

Let $U \subset \mathbb{C}$ be a simply connected domain, $f : U \to \mathbb{C}$ nowhere zero and holomorphic, and $k \in \mathbb{N}$. Then there exists a holomorphic $g : U \to \mathbb{C}$ such that

$$\left(g(z)\right)^k = f(z).$$

If we have the logarithm, we can take roots.

Corollary

Let $U \subset \mathbb{C}$ be a simply connected domain, $f \colon U \to \mathbb{C}$ nowhere zero and holomorphic, and $k \in \mathbb{N}$. Then there exists a holomorphic $g \colon U \to \mathbb{C}$ such that

$$\big(g(z)\big)^k = f(z).$$

Proof: Find a $\psi \colon U \to \mathbb{C}$ such that $e^{\psi(z)} = f(z)$.

If we have the logarithm, we can take roots.

Corollary

Let $U \subset \mathbb{C}$ be a simply connected domain, $f : U \to \mathbb{C}$ nowhere zero and holomorphic, and $k \in \mathbb{N}$. Then there exists a holomorphic $g : U \to \mathbb{C}$ such that

$$\big(g(z)\big)^k = f(z).$$

Proof: Find a $\psi : U \to \mathbb{C}$ such that $e^{\psi(z)} = f(z)$.

Let $g(z) = e^{\frac{1}{k}\psi(z)}$.

If we have the logarithm, we can take roots.

Corollary

Let $U \subset \mathbb{C}$ be a simply connected domain, $f : U \to \mathbb{C}$ nowhere zero and holomorphic, and $k \in \mathbb{N}$. Then there exists a holomorphic $g : U \to \mathbb{C}$ such that

$$\big(g(z)\big)^k = f(z).$$

Proof: Find a $\psi : U \to \mathbb{C}$ such that $e^{\psi(z)} = f(z)$.

Let $g(z) = e^{\frac{1}{k}\psi(z)}$.

Check: $\qquad \big(g(z)\big)^k$

If we have the logarithm, we can take roots.

Corollary

Let $U \subset \mathbb{C}$ be a simply connected domain, $f\colon U \to \mathbb{C}$ nowhere zero and holomorphic, and $k \in \mathbb{N}$. Then there exists a holomorphic $g\colon U \to \mathbb{C}$ such that

$$\big(g(z)\big)^k = f(z).$$

Proof: Find a $\psi\colon U \to \mathbb{C}$ such that $e^{\psi(z)} = f(z)$.

Let $g(z) = e^{\frac{1}{k}\psi(z)}$.

Check: $$\big(g(z)\big)^k = \left(e^{\frac{1}{k}\psi(z)}\right)^k$$

If we have the logarithm, we can take roots.

Corollary

Let $U \subset \mathbb{C}$ be a simply connected domain, $f : U \to \mathbb{C}$ nowhere zero and holomorphic, and $k \in \mathbb{N}$. Then there exists a holomorphic $g : U \to \mathbb{C}$ such that

$$\big(g(z)\big)^k = f(z).$$

Proof: Find a $\psi : U \to \mathbb{C}$ such that $e^{\psi(z)} = f(z)$.

Let $g(z) = e^{\frac{1}{k}\psi(z)}$.

Check: $$\big(g(z)\big)^k = \left(e^{\frac{1}{k}\psi(z)}\right)^k = e^{\psi(z)}$$

If we have the logarithm, we can take roots.

Corollary

Let $U \subset \mathbb{C}$ be a simply connected domain, $f : U \to \mathbb{C}$ nowhere zero and holomorphic, and $k \in \mathbb{N}$. Then there exists a holomorphic $g : U \to \mathbb{C}$ such that

$$\big(g(z)\big)^k = f(z).$$

Proof: Find a $\psi : U \to \mathbb{C}$ such that $e^{\psi(z)} = f(z)$.

Let $g(z) = e^{\frac{1}{k}\psi(z)}$.

Check: $$\big(g(z)\big)^k = \left(e^{\frac{1}{k}\psi(z)}\right)^k = e^{\psi(z)} = f(z).$$ □