# Cultivating Complex Analysis: Harmonic functions Harnack's inequality and principle (7.2.3–7.2.4)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

## Theorem (Harnack's inequality)

*Suppose* $f\colon \Delta_R(p) \to \mathbb{R}$ *is harmonic and nonnegative, and suppose* $0 < r < R$. *Then*

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As $S < R$ was arbitrary, the theorem follows by taking a limit. ◻

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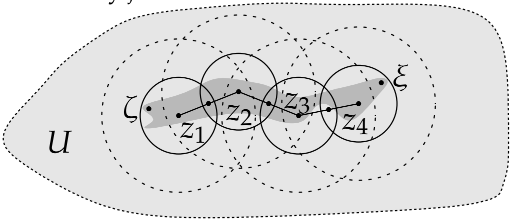

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**Exercise:** Use Harnack's inequality to prove Liouville's theorem for harmonic functions: If $f : \mathbb{C} \to \mathbb{R}$ is harmonic and nonnegative, then $f$ is constant.

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$$\sup_{z \in K} \bigl(f_m(z) - f_n(z)\bigr) \leq C \inf_{z \in K} \bigl(f_m(z) - f_n(z)\bigr) \leq C\bigl(f_m(p) - f_n(p)\bigr) < \epsilon.$$

$\Rightarrow \quad \{f_n\}$ is uniformly Cauchy on $K \quad \Rightarrow \quad$ converges uniformly.
$f$ is harmonic by Harnack's first theorem. $\square$

**Exercise:** Prove yet another version of Harnack's principle. Suppose $U \subset \mathbb{C}$ is a domain, $\{f_n\}$ is a sequence of nonnegative harmonic functions on $U$, and $p \in U$ is fixed.

a) If $f_n(p) \to +\infty$, then $\{f_n\}$ converges to $+\infty$ uniformly on compact subsets.

b) If $f: U \to \mathbb{R}$ is harmonic, $f_n(z) \leq f(z)$ for all $z \in U$, and $f_n(p) \to f(p)$, then $\{f_n\}$ converges to $f$ uniformly on compact subsets.

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**Exercise:** Prove a Montel-like theorem for harmonic functions. Suppose $U \subset \mathbb{C}$ is a domain and $\{f_n\}$ is a sequence of nonnegative harmonic functions. Show that at least one (or both) of the following are true:

(i) $\exists$ a subsequence converging to $+\infty$ uniformly on compact subsets.

(ii) $\exists$ a subsequence converging to a harmonic function uniformly on compact subsets.