# Cultivating Complex Analysis: The exponential (1.2.1) Polar coordinates (1.2.2)

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Define the exponential $e^z$ for $z \in \mathbb{C}$ (using the real exponential and sin/cos):

$$\exp(z) = e^z = e^{x+iy} \stackrel{\text{def}}{=} e^x e^{iy} = e^x \cos y + i e^x \sin y.$$

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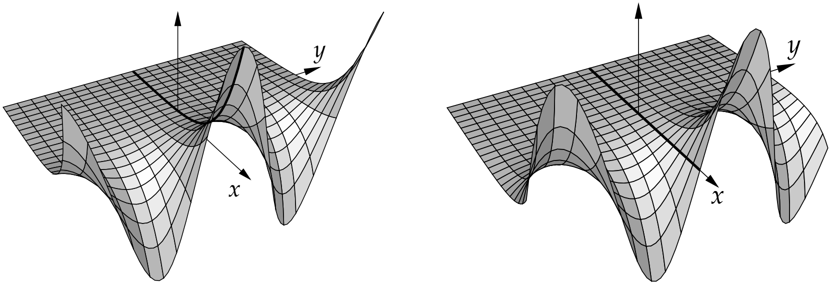
Graphs of the real and imaginary part:



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We define sin and cos for $z \in \mathbb{C}$ accordingly:

$$\cos z \stackrel{\text{def}}{=} \frac{e^{iz} + e^{-iz}}{2}, \qquad \sin z \stackrel{\text{def}}{=} \frac{e^{iz} - e^{-iz}}{2i}.$$

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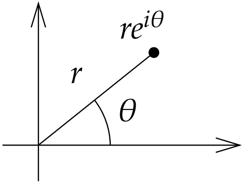

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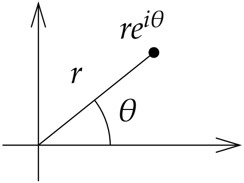


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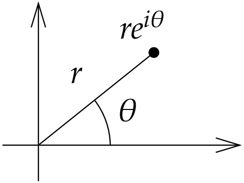


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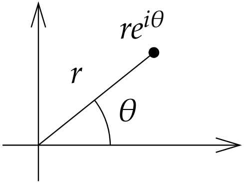


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The downside is that the polar form is terrible for addition.