

Cultivating Complex Analysis: The complex numbers as the plane (1.1.1)

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Start with \mathbb{R} and arrive at \mathbb{C} to be able to solve $z^2 + 1 = 0$.

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A graduate complex analysis course fills the student with unrealistic optimism.

Definition of the complex numbers (complex field, complex plane):

$$\mathbb{C} \stackrel{\text{def}}{=} \mathbb{R}^2$$

with multiplication:

$$(a, b) + (c, d) \stackrel{\text{def}}{=} (a + c, b + d),$$
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$i^2 = -1$ so $z^2 + 1 = 0$ has the two solutions i and $-i$.

$(x, y) = x + iy$, so use $x + iy$ from now on (*cartesian form*).

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Let's give names to the x and y :

$$\operatorname{Re} z = \operatorname{Re}(x + iy) \stackrel{\text{def}}{=} \frac{z + \bar{z}}{2} = x \quad \text{real part of } z.$$

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An expression in x, y can be written in terms of z, \bar{z} and vice versa:

$$x^3 + y^3 + 3ixy = \left(\frac{z + \bar{z}}{2}\right)^3 + \left(\frac{z - \bar{z}}{2i}\right)^3 + 3i\left(\frac{z + \bar{z}}{2}\right)\left(\frac{z - \bar{z}}{2i}\right),$$

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$$z^2 - i\bar{z}^2 + z\bar{z} = (x + iy)^2 - i(x - iy)^2 + (x + iy)(x - iy).$$

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Almost looks as if z and \bar{z} were independent variables.