Cultivating Complex Analysis: Residue theorem, applications (5.3 part 2)

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Recall that the residue theorem says that given a finite $S \subset U$, Γ a cycle in $U \setminus S$ homologous to zero in U, and $f \colon U \setminus S \to \mathbb{C}$ holomorphic,

$$\frac{1}{2\pi i} \int_\Gamma f(z)\, dz = \sum_{p \in S} n(\Gamma; p) \operatorname{Res}(f; p),$$

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Because we have lots of tricks to compute c_{-1}. We'll go over a few.

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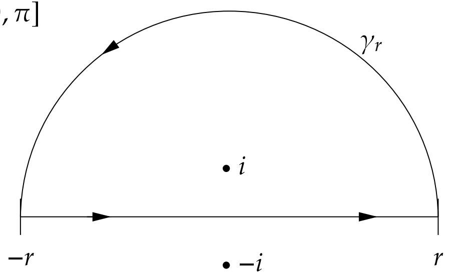

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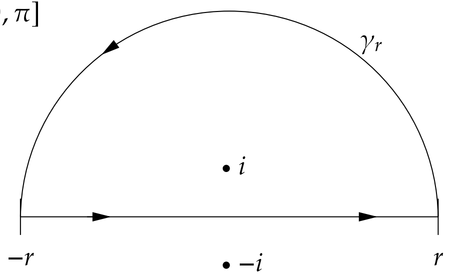

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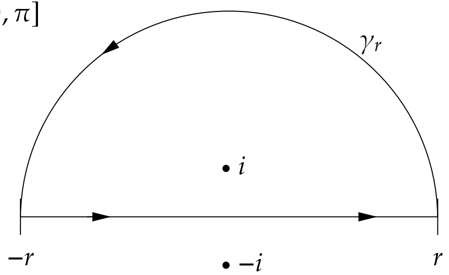

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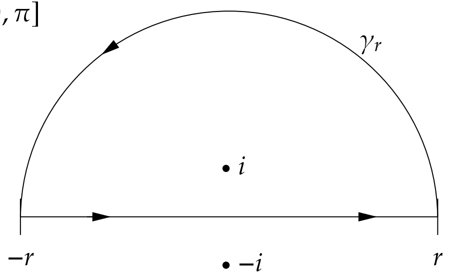

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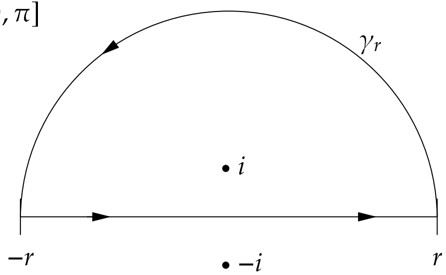

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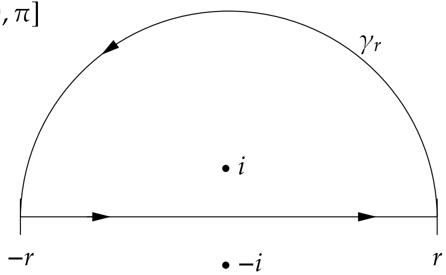

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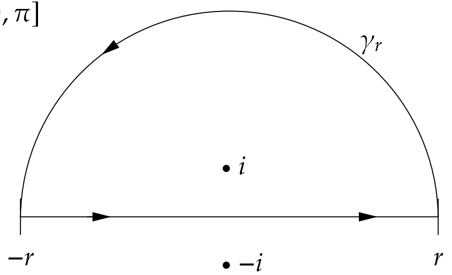

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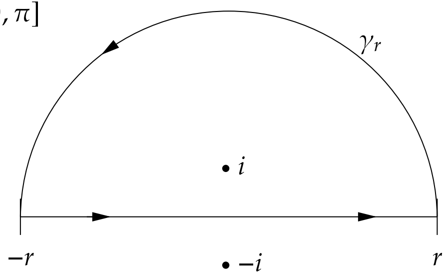

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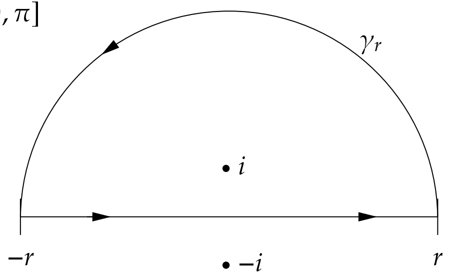

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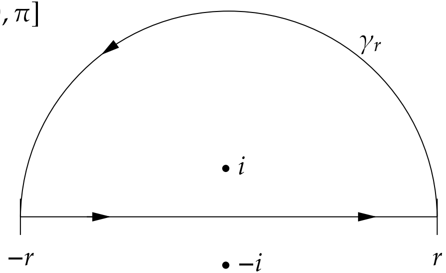

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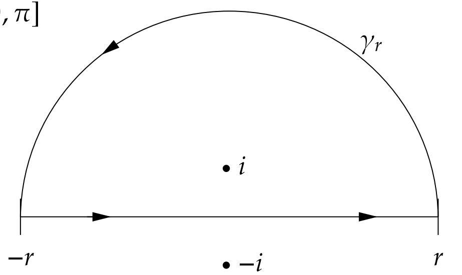

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Minor technicality: Why the symmetric limit is sufficient?

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On the unit circle $\bar{z} = {}^1\!/_z$. So if $z = e^{i\theta}$, $\cos\theta = \operatorname{Re} z = \frac{z+1/z}{2}$ and $\sin\theta = \operatorname{Im} z = \frac{z-1/z}{2i}$.

Example: Suppose $c > 1$.

$$\int_0^{2\pi} \frac{1}{c+\cos\theta}\, d\theta = \int_{\partial\mathbb{D}} \frac{1}{c+\frac{z+1/z}{2}} \frac{1}{iz}\, dz = -2i \int_{\partial\mathbb{D}} \frac{1}{z^2+2cz+1}\, dz.$$

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A common computation via the residue theorem are inverse Laplace transforms. *Mellin's inversion formula* says that given a transform $F(s)$, the original $f(t)$ is given by

$$f(t) = \mathcal{L}^{-1}\left[F(s)\right] = \frac{1}{2\pi i} \lim_{r\to\infty} \int_{c-ir}^{c+ir} e^{st} F(s)\, ds$$

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As an exercise, try your hand at computing a few. Say

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Hint: Pick the correct vertical line (pick a c) and an arc that goes around all the poles.