

Cultivating Complex Analysis: Holomorphic functions are analytic (3.3.1)

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Moreover,

$$c_n = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)}{(z - p)^{n+1}} dz,$$

where γ is any circle of radius r , $0 < r < R$, centered at p oriented counterclockwise.

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The last equality held because the sum converges uniformly in $\zeta \in \partial\Delta_r(p)$ (we'll justify that on the next slide).

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We get the same series for every r and it converges in $\Delta_R(p)$.



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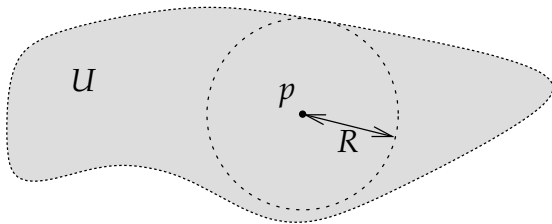
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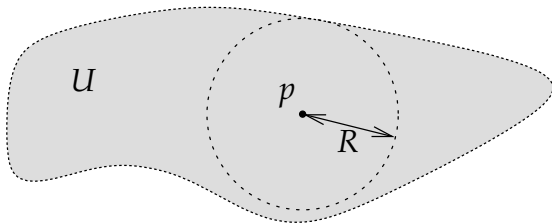
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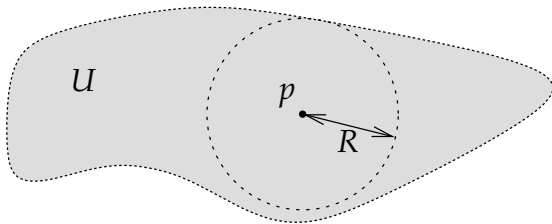
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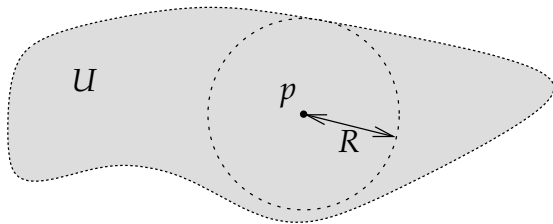
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Remark: Nothing like this is true for real-analytic functions such as $\varphi(x) = \frac{1}{1+x^2}$ whose radius of convergence at $x = 0$ is 1, but $\varphi: \mathbb{R} \rightarrow \mathbb{R}$ is (real) analytic everywhere.

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We have also finally proved the following:

A convergent power series defines an analytic function.