

Cultivating Complex Analysis: Riemann mapping theorem (6.3.1)

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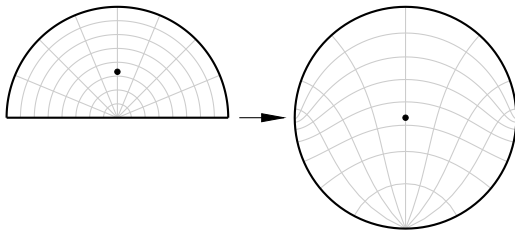
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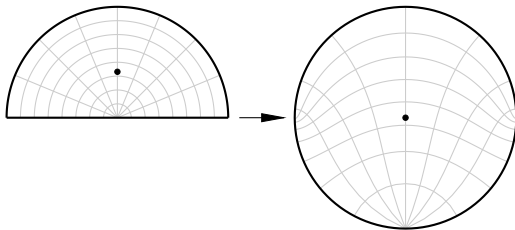
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Proof is to “maximize” $|f'(p)|$ among all maps into the disc and $f(p) = 0$.

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Differentiate $\varphi_q \circ f = g^2$ at p (recall $\varphi'_a(0) = 1 - |a|^2$):

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Uniqueness left as an exercise.



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The nicer the boundary, the nicer the map will be.

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Exercise: Suppose $f: \mathbb{C} \rightarrow \mathbb{C}$ is entire holomorphic and injective, prove that f is onto.