

# Cultivating Complex Analysis: The argument principle (5.4.1)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

**Terminology:** *zeros/poles counted with multiplicity:*  $f(z) = z^2(z - 1)^3$  has the zeros  $z_1, z_2, z_3, z_4, z_5 = 0, 0, 1, 1, 1$ .

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### Theorem (Argument principle)

Suppose  $U \subset \mathbb{C}$  is open and  $\Gamma$  is a cycle in  $U$  homologous to zero in  $U$ . Suppose  $f: U \rightarrow \mathbb{C}_\infty$  is a meromorphic function with no zeros or poles on  $\Gamma$ . Let  $z_1, \dots, z_n$  denote the zeros of  $f$  counted with multiplicity, and let  $p_1, \dots, p_\ell$  denote the poles of  $f$  counted with multiplicity. Then

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} dz = \sum_{k=1}^n n(\Gamma; z_k) - \sum_{k=1}^{\ell} n(\Gamma; p_k).$$

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# of poles/zero normally countable, but can assume finite above.

Suppose  $n(\Gamma; z) = 1$  or  $0$  for all  $z \in U$ .

The “inside of  $\Gamma$ ” are the points where  $n(\Gamma; z) = 1$ .

If there are  $n$  zeros and  $\ell$  poles (counting multiplicity) inside  $\Gamma$ , then

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Another interpretation:

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i} \int_{f \circ \gamma} \frac{1}{\zeta} d\zeta = n(f \circ \gamma; 0).$$



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WLOG suppose it is the origin.

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Everything except  $m h(0) \frac{1}{z}$  is holomorphic. So

$$\operatorname{Res} \left( h \frac{f'}{f}; 0 \right) = m h(0) \quad \square$$



Application: Locate zeros of holomorphic  $f$  (e.g. polynomials) by computing (even numerically)

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Related application:

If  $z_1, \dots, z_n$  are zeros of  $f$  inside  $\Gamma$  (going around them once), then

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If there is one simple zero  $z_0$  of  $f$  within  $\Gamma$ , then

$$\frac{1}{2\pi i} \int_{\Gamma} z \frac{f'(z)}{f(z)} dz = z_0.$$