Cultivating Complex Analysis: The maximum modulus principle (3.3.3)

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Take a closed disc $\overline{\Delta_r(0)} \subset U$,
where r is small enough so that $|f(z)| \leq |f(0)| = f(0)$ whenever $|z| \leq r$.

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$$0 = \mathrm{Re} \int_0^{2\pi} \Big(|f(re^{it})| - f(re^{it}) \Big)\, dt = \int_0^{2\pi} \Big(|f(re^{it})| - \mathrm{Re}\, f(re^{it}) \Big)\, dt.$$

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Proof is an exercise.

There's a version for a miminum if you avoid zeros:

Exercise: (Minimum modulus principle) Suppose $U \subset \mathbb{C}$ is a domain and $f\colon U \to \mathbb{C}$ is holomorphic. If $|f(z)|$ achieves a local minimum at $p \in U$ and $f(p) \neq 0$, then f is constant.