Cultivating Complex Analysis: Montel's theorem (6.2)

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Definition

Let $U \subset \mathbb{C}$ be open. A set $\mathcal{F}$ of holomorphic functions $f : U \to \mathbb{C}$ is called a *normal family* if every sequence in $\mathcal{F}$ has a subsequence that converges uniformly on compact subsets (the limit need not be in $\mathcal{F}$).

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Exercise: Prove that "locally bounded" means "bounded on compact subsets."

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Another commonly used consequence of Montel is Vitali's theorem.

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Suppose $U \subset \mathbb{C}$ is a domain, $\{f_n\}$ is a locally bounded sequence of holomorphic functions that converges pointwise on a set $E \subset U$, and E has a limit point in U. Then $\{f_n\}$ converges uniformly on compact subsets in U.

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Proof is an exercise.

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Exercise: Show that if the partial sums of a power series centered at p are uniformly bounded on $\Delta_r(p)$ for some $r > 0$, then the power series converges in $\Delta_r(p)$.