Cultivating Complex Analysis:
Convergence of subsequences (6.1.1)
Equicontinuity (6.1.2)

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Example:
$\dfrac{n^2 x}{1 + n^2 x^2}$ is pointwise bounded (converges pointwise) but not uniformly bounded.

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For any m, the tail $\{f_{k,k}\}_{k=m}^{\infty}$ is a subsequence of $\{f_{m,k}\}_{k=1}^{\infty}$ $\quad\Rightarrow\quad \{f_{k,k}(x_m)\}_{k=1}^{\infty}$ converges. $\square$

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Definition

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For finite sets S, same as continuity and uniform continuity.

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Proof: Suppose $\{f_n\}$ is not uniformly equicontinuous.
$\Rightarrow\ \exists\, \epsilon > 0$ s.t. $\forall k \in \mathbb{N}$, $\exists\, n_k \in \mathbb{N}$ & $x_k, y_k \in X$ with $d(x_k, y_k) < 1/k$ where $|f_{n_k}(x_k) - f_{n_k}(y_k)| \geq \epsilon$.

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$\Rightarrow$ $\{f_n\}$ is not equicontinuous at x. $\square$

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Exercise: Suppose S is a set of (real) differentiable functions $f\colon [0,1] \to \mathbb{R}$ such that $|f'(x)| \leq 1$ for all $x \in [0,1]$. Prove that S is uniformly equicontinuous.