Cultivating Complex Analysis: Cycles around compacts (6.3.3)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

Given a compact $K \subset U$, we want a Γ in $U \setminus K$ homologous to zero in U that goes around K.

Given a compact $K \subset U$, we want a Γ in $U \setminus K$ homologous to zero in U that goes around K.

Lemma

Let $U \subset \mathbb{C}$ be open and suppose that $K \subset U$ is compact and nonempty. Then there exists a cycle Γ in $U \setminus K$ such that $n(\Gamma; z) = 1$ for all $z \in K$ and $n(\Gamma; z) = 0$ for all $z \in \mathbb{C} \setminus U$ and such that $n(\Gamma; z)$ is 0 or 1 for all $z \notin \Gamma$.

Given a compact $K \subset U$, we want a Γ in $U \setminus K$ homologous to zero in U that goes around K.

Lemma

Let $U \subset \mathbb{C}$ be open and suppose that $K \subset U$ is compact and nonempty. Then there exists a cycle Γ in $U \setminus K$ such that $n(\Gamma; z) = 1$ for all $z \in K$ and $n(\Gamma; z) = 0$ for all $z \in \mathbb{C} \setminus U$ and such that $n(\Gamma; z)$ is 0 or 1 for all $z \notin \Gamma$.

Lots of ideas how to do it, but proof always involves checking many details.

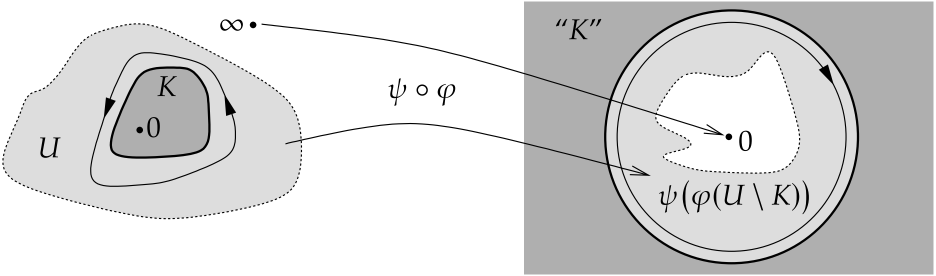
Given a compact $K \subset U$, we want a Γ in $U \setminus K$ homologous to zero in U that goes around K.

Lemma

Let $U \subset \mathbb{C}$ be open and suppose that $K \subset U$ is compact and nonempty. Then there exists a cycle Γ in $U \setminus K$ such that $n(\Gamma; z) = 1$ for all $z \in K$ and $n(\Gamma; z) = 0$ for all $z \in \mathbb{C} \setminus U$ and such that $n(\Gamma; z)$ is 0 or 1 for all $z \notin \Gamma$.

Lots of ideas how to do it, but proof always involves checking many details.

We will map to the disk, but with a twist. We'll take $\mathbb{C}_\infty \setminus K$ to the disc, and go around the "outside" in the opposite direction:



Proof: K could have infinitely many components.

Proof: K could have infinitely many components. For a small $r > 0$, $\exists$ closed discs such that

$$K \subset K' = \overline{\Delta_r(z_1)} \cup \cdots \cup \overline{\Delta_r(z_m)} \subset U.$$

K' is compact and has only finitely many components.

Proof: K could have infinitely many components. For a small $r > 0$, $\exists$ closed discs such that

$$K \subset K' = \overline{\Delta_r(z_1)} \cup \cdots \cup \overline{\Delta_r(z_m)} \subset U.$$

K' is compact and has only finitely many components. A Γ around K' suffices as $K \subset K'$.

Proof: K could have infinitely many components. For a small $r > 0$, $\exists$ closed discs such that

$$K \subset K' = \overline{\Delta_r(z_1)} \cup \cdots \cup \overline{\Delta_r(z_m)} \subset U.$$

K' is compact and has only finitely many components. A Γ around K' suffices as $K \subset K'$.

Let $K_1, \ldots, K_n$ be the components of K'.

Proof: K could have infinitely many components. For a small $r > 0$, $\exists$ closed discs such that

$$K \subset K' = \overline{\Delta_r(z_1)} \cup \cdots \cup \overline{\Delta_r(z_m)} \subset U.$$

K' is compact and has only finitely many components. A Γ around K' suffices as $K \subset K'$.

Let $K_1, \ldots, K_n$ be the components of K'. K_1 and $K_2 \cup \cdots \cup K_n$ are closed.

Proof: K could have infinitely many components. For a small $r > 0$, $\exists$ closed discs such that

$$K \subset K' = \overline{\Delta_r(z_1)} \cup \cdots \cup \overline{\Delta_r(z_m)} \subset U.$$

K' is compact and has only finitely many components. A Γ around K' suffices as $K \subset K'$.

Let $K_1, \ldots, K_n$ be the components of K'. $\quad K_1$ and $K_2 \cup \cdots \cup K_n$ are closed.

If we prove the lemma for K_1 and $U \setminus (K_2 \cup \cdots \cup K_n)$ to find a cycle Γ_1, then we are done:

Proof: K could have infinitely many components. For a small $r > 0$, $\exists$ closed discs such that

$$K \subset K' = \overline{\Delta_r(z_1)} \cup \cdots \cup \overline{\Delta_r(z_m)} \subset U.$$

K' is compact and has only finitely many components. A Γ around K' suffices as $K \subset K'$.

Let $K_1, \ldots, K_n$ be the components of K'. K_1 and $K_2 \cup \cdots \cup K_n$ are closed.

If we prove the lemma for K_1 and $U \setminus (K_2 \cup \cdots \cup K_n)$ to find a cycle Γ_1, then we are done:

Repeat the procedure for each K_j to find Γ_j and let $\Gamma = \Gamma_1 + \cdots + \Gamma_n$.

Proof: K could have infinitely many components. For a small $r > 0$, $\exists$ closed discs such that

$$K \subset K' = \overline{\Delta_r(z_1)} \cup \cdots \cup \overline{\Delta_r(z_m)} \subset U.$$

K' is compact and has only finitely many components. A Γ around K' suffices as $K \subset K'$.

Let $K_1, \ldots, K_n$ be the components of K'. K_1 and $K_2 \cup \cdots \cup K_n$ are closed.

If we prove the lemma for K_1 and $U \setminus (K_2 \cup \cdots \cup K_n)$ to find a cycle Γ_1, then we are done:

Repeat the procedure for each K_j to find Γ_j and let $\Gamma = \Gamma_1 + \cdots + \Gamma_n$.

$n(\Gamma_j; z) = 1$ for all $z \in K_j$

Proof: K could have infinitely many components. For a small $r > 0$, $\exists$ closed discs such that

$$K \subset K' = \overline{\Delta_r(z_1)} \cup \cdots \cup \overline{\Delta_r(z_m)} \subset U.$$

K' is compact and has only finitely many components. A Γ around K' suffices as $K \subset K'$.

Let $K_1, \ldots, K_n$ be the components of K'. K_1 and $K_2 \cup \cdots \cup K_n$ are closed.

If we prove the lemma for K_1 and $U \setminus (K_2 \cup \cdots \cup K_n)$ to find a cycle Γ_1, then we are done:

Repeat the procedure for each K_j to find Γ_j and let $\Gamma = \Gamma_1 + \cdots + \Gamma_n$.

$n(\Gamma_j; z) = 1$ for all $z \in K_j$ and $n(\Gamma_j; z) = 0$ for all $z \in K_\ell$ if $\ell \neq j$.

Proof: K could have infinitely many components. For a small $r > 0$, $\exists$ closed discs such that

$$K \subset K' = \overline{\Delta_r(z_1)} \cup \cdots \cup \overline{\Delta_r(z_m)} \subset U.$$

K' is compact and has only finitely many components. A Γ around K' suffices as $K \subset K'$.

Let $K_1, \ldots, K_n$ be the components of K'. K_1 and $K_2 \cup \cdots \cup K_n$ are closed.

If we prove the lemma for K_1 and $U \setminus (K_2 \cup \cdots \cup K_n)$ to find a cycle Γ_1, then we are done:

Repeat the procedure for each K_j to find Γ_j and let $\Gamma = \Gamma_1 + \cdots + \Gamma_n$.

$n(\Gamma_j; z) = 1$ for all $z \in K_j$ and $n(\Gamma_j; z) = 0$ for all $z \in K_\ell$ if $\ell \neq j$. So Γ works.

Proof: K could have infinitely many components. For a small $r > 0$, $\exists$ closed discs such that

$$K \subset K' = \overline{\Delta_r(z_1)} \cup \cdots \cup \overline{\Delta_r(z_m)} \subset U.$$

K' is compact and has only finitely many components. A Γ around K' suffices as $K \subset K'$.

Let $K_1, \ldots, K_n$ be the components of K'. K_1 and $K_2 \cup \cdots \cup K_n$ are closed.

If we prove the lemma for K_1 and $U \setminus (K_2 \cup \cdots \cup K_n)$ to find a cycle Γ_1, then we are done:

Repeat the procedure for each K_j to find Γ_j and let $\Gamma = \Gamma_1 + \cdots + \Gamma_n$.

$n(\Gamma_j; z) = 1$ for all $z \in K_j$ and $n(\Gamma_j; z) = 0$ for all $z \in K_\ell$ if $\ell \neq j$. So Γ works.

So without loss of generality, assume that K is connected.

Assume $0 \in K$.

Assume $0 \in K$. Assume K has more than one point.

Assume $0 \in K$. Assume K has more than one point.

Let $\varphi\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, be $\varphi(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$, $\varphi(0) = \infty$ and $\varphi(\infty) = 0$.

Assume $0 \in K$. Assume K has more than one point.

Let $\varphi\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, be $\varphi(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$, $\varphi(0) = \infty$ and $\varphi(\infty) = 0$. Let

$$V = \varphi(\mathbb{C}_\infty \setminus K).$$

Assume $0 \in K$. Assume K has more than one point.

Let $\varphi \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, be $\varphi(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$, $\varphi(0) = \infty$ and $\varphi(\infty) = 0$. Let

$$V = \varphi(\mathbb{C}_\infty \setminus K).$$

$\infty \notin V$,

Assume $0 \in K$. Assume K has more than one point.

Let $\varphi\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, be $\varphi(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$, $\varphi(0) = \infty$ and $\varphi(\infty) = 0$. Let

$$V = \varphi(\mathbb{C}_\infty \setminus K).$$

$\infty \notin V, 0 \in V,$

Assume $0 \in K$. Assume K has more than one point.

Let $\varphi \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, be $\varphi(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$, $\varphi(0) = \infty$ and $\varphi(\infty) = 0$. Let

$$V = \varphi(\mathbb{C}_\infty \setminus K).$$

$\infty \notin V, 0 \in V, V \neq \mathbb{C},$

Assume $0 \in K$. Assume K has more than one point.

Let $\varphi \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, be $\varphi(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$, $\varphi(0) = \infty$ and $\varphi(\infty) = 0$. Let

$$V = \varphi(\mathbb{C}_\infty \setminus K).$$

$\infty \notin V$, $0 \in V$, $V \neq \mathbb{C}$, and $\mathbb{C}_\infty \setminus V = \varphi(K)$ is connected.

Assume $0 \in K$. Assume K has more than one point.

Let $\varphi \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, be $\varphi(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$, $\varphi(0) = \infty$ and $\varphi(\infty) = 0$. Let

$$V = \varphi(\mathbb{C}_\infty \setminus K).$$

$\infty \notin V$, $0 \in V$, $V \neq \mathbb{C}$, and $\mathbb{C}_\infty \setminus V = \varphi(K)$ is connected.
So components of V are simply connected (exercise).

Assume $0 \in K$. Assume K has more than one point.

Let $\varphi \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, be $\varphi(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$, $\varphi(0) = \infty$ and $\varphi(\infty) = 0$. Let

$$V = \varphi(\mathbb{C}_\infty \setminus K).$$

$\infty \notin V$, $0 \in V$, $V \neq \mathbb{C}$, and $\mathbb{C}_\infty \setminus V = \varphi(K)$ is connected.
So components of V are simply connected (exercise).
K a union of discs $\quad\Rightarrow\quad \mathbb{C}_\infty \setminus K$ and thus V has finitely many components $V_1, \ldots, V_m$.

Assume $0 \in K$. Assume K has more than one point.

Let $\varphi \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, be $\varphi(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$, $\varphi(0) = \infty$ and $\varphi(\infty) = 0$. Let

$$V = \varphi(\mathbb{C}_\infty \setminus K).$$

$\infty \notin V$, $0 \in V$, $V \neq \mathbb{C}$, and $\mathbb{C}_\infty \setminus V = \varphi(K)$ is connected.
So components of V are simply connected (exercise).
K a union of discs $\quad \Rightarrow \quad \mathbb{C}_\infty \setminus K$ and thus V has finitely many components $V_1, \ldots, V_m$.

By RMT, $\forall j$, $\exists$ a biholomorphic map from V_j to $\Delta_1(q_j)$ (disjoint).

Assume $0 \in K$. Assume K has more than one point.

Let $\varphi \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, be $\varphi(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$, $\varphi(0) = \infty$ and $\varphi(\infty) = 0$. Let

$$V = \varphi(\mathbb{C}_\infty \setminus K).$$

$\infty \notin V$, $0 \in V$, $V \neq \mathbb{C}$, and $\mathbb{C}_\infty \setminus V = \varphi(K)$ is connected.
So components of V are simply connected (exercise).
K a union of discs $\quad \Rightarrow \quad \mathbb{C}_\infty \setminus K$ and thus V has finitely many components $V_1, \ldots, V_m$.

By RMT, $\forall j$, $\exists$ a biholomorphic map from V_j to $\Delta_1(q_j)$ (disjoint).
Write

$$D = \Delta_1(q_1) \cup \cdots \cup \Delta_1(q_m).$$

Assume $0 \in K$. Assume K has more than one point.

Let $\varphi \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, be $\varphi(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$, $\varphi(0) = \infty$ and $\varphi(\infty) = 0$. Let

$$V = \varphi(\mathbb{C}_\infty \setminus K).$$

$\infty \notin V$, $0 \in V$, $V \neq \mathbb{C}$, and $\mathbb{C}_\infty \setminus V = \varphi(K)$ is connected.
So components of V are simply connected (exercise).
K a union of discs $\quad \Rightarrow \quad \mathbb{C}_\infty \setminus K$ and thus V has finitely many components $V_1, \dots, V_m$.

By RMT, $\forall j$, $\exists$ a biholomorphic map from V_j to $\Delta_1(q_j)$ (disjoint).
Write

$$D = \Delta_1(q_1) \cup \cdots \cup \Delta_1(q_m).$$

So $\exists$ biholomorphic $\psi \colon V \to D$,

Assume $0 \in K$. Assume K has more than one point.

Let $\varphi\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, be $\varphi(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$, $\varphi(0) = \infty$ and $\varphi(\infty) = 0$. Let

$$V = \varphi(\mathbb{C}_\infty \setminus K).$$

$\infty \notin V$, $0 \in V$, $V \neq \mathbb{C}$, and $\mathbb{C}_\infty \setminus V = \varphi(K)$ is connected.
So components of V are simply connected (exercise).
K a union of discs $\quad\Rightarrow\quad \mathbb{C}_\infty \setminus K$ and thus V has finitely many components $V_1, \ldots, V_m$.

By RMT, $\forall j$, $\exists$ a biholomorphic map from V_j to $\Delta_1(q_j)$ (disjoint).
Write

$$D = \Delta_1(q_1) \cup \cdots \cup \Delta_1(q_m).$$

So $\exists$ biholomorphic $\psi\colon V \to D$, $q_1 = 0$ and $\psi(0) = 0 = q_1$.

Assume $0 \in K$. Assume K has more than one point.

Let $\varphi\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, be $\varphi(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$, $\varphi(0) = \infty$ and $\varphi(\infty) = 0$. Let

$$V = \varphi(\mathbb{C}_\infty \setminus K).$$

$\infty \notin V$, $0 \in V$, $V \neq \mathbb{C}$, and $\mathbb{C}_\infty \setminus V = \varphi(K)$ is connected.
So components of V are simply connected (exercise).
K a union of discs $\quad\Rightarrow\quad \mathbb{C}_\infty \setminus K$ and thus V has finitely many components $V_1, \ldots, V_m$.

By RMT, $\forall j$, $\exists$ a biholomorphic map from V_j to $\Delta_1(q_j)$ (disjoint).
Write

$$D = \Delta_1(q_1) \cup \cdots \cup \Delta_1(q_m).$$

So $\exists$ biholomorphic $\psi\colon V \to D$, $q_1 = 0$ and $\psi(0) = 0 = q_1$.
$\mathbb{C}_\infty \setminus U$ is compact

Assume $0 \in K$. Assume K has more than one point.

Let $\varphi \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, be $\varphi(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$, $\varphi(0) = \infty$ and $\varphi(\infty) = 0$. Let

$$V = \varphi(\mathbb{C}_\infty \setminus K).$$

$\infty \notin V$, $0 \in V$, $V \neq \mathbb{C}$, and $\mathbb{C}_\infty \setminus V = \varphi(K)$ is connected.
So components of V are simply connected (exercise).
K a union of discs $\quad \Rightarrow \quad \mathbb{C}_\infty \setminus K$ and thus V has finitely many components $V_1, \ldots, V_m$.

By RMT, $\forall j$, $\exists$ a biholomorphic map from V_j to $\Delta_1(q_j)$ (disjoint).
Write

$$D = \Delta_1(q_1) \cup \cdots \cup \Delta_1(q_m).$$

So $\exists$ biholomorphic $\psi \colon V \to D$, $q_1 = 0$ and $\psi(0) = 0 = q_1$.
$\mathbb{C}_\infty \setminus U$ is compact $\Rightarrow$ $\varphi(\mathbb{C}_\infty \setminus U) \subset V$ is compact

Assume $0 \in K$. Assume K has more than one point.

Let $\varphi\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, be $\varphi(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$, $\varphi(0) = \infty$ and $\varphi(\infty) = 0$. Let

$$V = \varphi(\mathbb{C}_\infty \setminus K).$$

$\infty \notin V$, $0 \in V$, $V \neq \mathbb{C}$, and $\mathbb{C}_\infty \setminus V = \varphi(K)$ is connected.
So components of V are simply connected (exercise).
K a union of discs $\quad \Rightarrow \quad \mathbb{C}_\infty \setminus K$ and thus V has finitely many components $V_1, \ldots, V_m$.

By RMT, $\forall j$, $\exists$ a biholomorphic map from V_j to $\Delta_1(q_j)$ (disjoint).
Write

$$D = \Delta_1(q_1) \cup \cdots \cup \Delta_1(q_m).$$

So $\exists$ biholomorphic $\psi\colon V \to D$, $q_1 = 0$ and $\psi(0) = 0 = q_1$.
$\mathbb{C}_\infty \setminus U$ is compact $\Rightarrow$ $\varphi(\mathbb{C}_\infty \setminus U) \subset V$ is compact $\Rightarrow$ $S = \psi\big(\varphi(\mathbb{C}_\infty \setminus U)\big) \subset D$ is compact.

Assume $0 \in K$. Assume K has more than one point.

Let $\varphi\colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, be $\varphi(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$, $\varphi(0) = \infty$ and $\varphi(\infty) = 0$. Let

$$V = \varphi(\mathbb{C}_\infty \setminus K).$$

$\infty \notin V$, $0 \in V$, $V \neq \mathbb{C}$, and $\mathbb{C}_\infty \setminus V = \varphi(K)$ is connected.
So components of V are simply connected (exercise).
K a union of discs $\quad\Rightarrow\quad \mathbb{C}_\infty \setminus K$ and thus V has finitely many components $V_1, \dots, V_m$.

By RMT, $\forall j$, $\exists$ a biholomorphic map from V_j to $\Delta_1(q_j)$ (disjoint).
Write

$$D = \Delta_1(q_1) \cup \cdots \cup \Delta_1(q_m).$$

So $\exists$ biholomorphic $\psi\colon V \to D$, $q_1 = 0$ and $\psi(0) = 0 = q_1$.
$\mathbb{C}_\infty \setminus U$ is compact $\Rightarrow$ $\varphi(\mathbb{C}_\infty \setminus U) \subset V$ is compact $\Rightarrow$ $S = \psi\big(\varphi(\mathbb{C}_\infty \setminus U)\big) \subset D$ is compact.
$\exists\, r < 1$ such that

$$S \subset \Delta_r(q_1) \cup \cdots \cup \Delta_r(q_m)$$

Assume $0 \in K$. Assume K has more than one point.

Let $\varphi \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, be $\varphi(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$, $\varphi(0) = \infty$ and $\varphi(\infty) = 0$. Let

$$V = \varphi(\mathbb{C}_\infty \setminus K).$$

$\infty \notin V$, $0 \in V$, $V \neq \mathbb{C}$, and $\mathbb{C}_\infty \setminus V = \varphi(K)$ is connected.
So components of V are simply connected (exercise).
K a union of discs $\Rightarrow$ $\mathbb{C}_\infty \setminus K$ and thus V has finitely many components $V_1, \ldots, V_m$.

By RMT, $\forall j$, $\exists$ a biholomorphic map from V_j to $\Delta_1(q_j)$ (disjoint).
Write

$$D = \Delta_1(q_1) \cup \cdots \cup \Delta_1(q_m).$$

So $\exists$ biholomorphic $\psi \colon V \to D$, $q_1 = 0$ and $\psi(0) = 0 = q_1$.
$\mathbb{C}_\infty \setminus U$ is compact $\Rightarrow$ $\varphi(\mathbb{C}_\infty \setminus U) \subset V$ is compact $\Rightarrow$ $S = \psi\big(\varphi(\mathbb{C}_\infty \setminus U)\big) \subset D$ is compact.
$\exists\, r < 1$ such that

$$S \subset \Delta_r(q_1) \cup \cdots \cup \Delta_r(q_m)$$

Let $\gamma_j(t) = q_j + re^{-it}$ for $t \in [0, 2\pi]$ $\quad (\gamma_j = -\partial\Delta_r(q_j))$.

Assume $0 \in K$. Assume K has more than one point.

Let $\varphi \colon \mathbb{C}_\infty \to \mathbb{C}_\infty$, be $\varphi(z) = \frac{1}{z}$ for $z \in \mathbb{C} \setminus \{0\}$, $\varphi(0) = \infty$ and $\varphi(\infty) = 0$. Let

$$V = \varphi(\mathbb{C}_\infty \setminus K).$$

$\infty \notin V$, $0 \in V$, $V \neq \mathbb{C}$, and $\mathbb{C}_\infty \setminus V = \varphi(K)$ is connected.
So components of V are simply connected (exercise).
K a union of discs $\quad \Rightarrow \quad \mathbb{C}_\infty \setminus K$ and thus V has finitely many components $V_1, \ldots, V_m$.

By RMT, $\forall j$, $\exists$ a biholomorphic map from V_j to $\Delta_1(q_j)$ (disjoint).
Write

$$D = \Delta_1(q_1) \cup \cdots \cup \Delta_1(q_m).$$

So $\exists$ biholomorphic $\psi \colon V \to D$, $q_1 = 0$ and $\psi(0) = 0 = q_1$.
$\mathbb{C}_\infty \setminus U$ is compact $\Rightarrow$ $\varphi(\mathbb{C}_\infty \setminus U) \subset V$ is compact $\Rightarrow$ $S = \psi\big(\varphi(\mathbb{C}_\infty \setminus U)\big) \subset D$ is compact.
$\exists\, r < 1$ such that

$$S \subset \Delta_r(q_1) \cup \cdots \cup \Delta_r(q_m)$$

Let $\gamma_j(t) = q_j + re^{-it}$ for $t \in [0, 2\pi]$ $\quad (\gamma_j = -\partial\Delta_r(q_j))$.
Let $\Gamma_j = \varphi^{-1} \circ \psi^{-1} \circ \gamma_j$, and $\Gamma = \Gamma_1 + \cdots + \Gamma_m$.

Suppose $p \notin \Gamma$.

$$n(\Gamma;p) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\varphi^{-1}\circ\psi^{-1}\circ\gamma_j} \frac{1}{z-p}\, dz$$

Suppose $p \notin \Gamma$.

$$n(\Gamma;p) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\varphi^{-1}\circ\psi^{-1}\circ\gamma_j} \frac{1}{z-p}\,dz = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\psi^{-1}\circ\gamma_j} \frac{-1}{(1-\zeta p)\zeta}\,d\zeta$$

Suppose $p \notin \Gamma$.

$$n(\Gamma; p) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\varphi^{-1}\circ\psi^{-1}\circ\gamma_j} \frac{1}{z-p}\, dz = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\psi^{-1}\circ\gamma_j} \frac{-1}{(1-\zeta p)\zeta}\, d\zeta$$

$$= \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} \frac{-1}{\left(1-\psi^{-1}(\xi)p\right) \psi^{-1}(\xi)\, \psi'\left(\psi^{-1}(\xi)\right)}\, d\xi.$$

Suppose $p \notin \Gamma$.

$$n(\Gamma;p) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\varphi^{-1}\circ\psi^{-1}\circ\gamma_j} \frac{1}{z-p}\, dz = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\psi^{-1}\circ\gamma_j} \frac{-1}{(1-\zeta p)\zeta}\, d\zeta$$

$$= \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} \frac{-1}{\big(1-\psi^{-1}(\xi)p\big)\, \psi^{-1}(\xi)\, \psi'\big(\psi^{-1}(\xi)\big)}\, d\xi.$$

Suppose $p \in \mathbb{C} \setminus U$.

Suppose $p \notin \Gamma$.

$$n(\Gamma;p) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\varphi^{-1}\circ\psi^{-1}\circ\gamma_j} \frac{1}{z-p}\,dz = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\psi^{-1}\circ\gamma_j} \frac{-1}{(1-\zeta p)\zeta}\,d\zeta$$

$$= \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} \frac{-1}{\left(1-\psi^{-1}(\xi)p\right)\psi^{-1}(\xi)\,\psi'\left(\psi^{-1}(\xi)\right)}\,d\xi.$$

Suppose $p \in \mathbb{C} \setminus U$.

$$h(\xi) = \frac{-1}{\left(1-\psi^{-1}(\xi)p\right)\psi^{-1}(\xi)\,\psi'\left(\psi^{-1}(\xi)\right)}$$

has two (simple) poles: one at $\psi\left(\frac{1}{p}\right)$ and one at $q_1 = 0$. (third factor never zero)

Suppose $p \notin \Gamma$.

$$n(\Gamma; p) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\varphi^{-1}\circ\psi^{-1}\circ\gamma_j} \frac{1}{z-p}\, dz = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\psi^{-1}\circ\gamma_j} \frac{-1}{(1-\zeta p)\zeta}\, d\zeta$$

$$= \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} \frac{-1}{\left(1-\psi^{-1}(\xi)p\right)\psi^{-1}(\xi)\,\psi'\left(\psi^{-1}(\xi)\right)}\, d\xi.$$

Suppose $p \in \mathbb{C} \setminus U$.

$$h(\xi) = \frac{-1}{\left(1-\psi^{-1}(\xi)p\right)\psi^{-1}(\xi)\,\psi'\left(\psi^{-1}(\xi)\right)}$$

has two (simple) poles: one at $\psi\left(\frac{1}{p}\right)$ and one at $q_1 = 0$. (third factor never zero)

$$\operatorname{Res}(h; 0) = \frac{-1}{\left(1-\psi^{-1}(0)p\right)\psi'\left(\psi^{-1}(0)\right)} \frac{1}{\frac{1}{\psi'(\psi^{-1}(0))}} = -1$$

Suppose $p \notin \Gamma$.

$$n(\Gamma; p) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\varphi^{-1}\circ\psi^{-1}\circ\gamma_j} \frac{1}{z-p}\, dz = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\psi^{-1}\circ\gamma_j} \frac{-1}{(1-\zeta p)\zeta}\, d\zeta$$

$$= \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} \frac{-1}{\left(1-\psi^{-1}(\xi)p\right)\psi^{-1}(\xi)\,\psi'\left(\psi^{-1}(\xi)\right)}\, d\xi.$$

Suppose $p \in \mathbb{C} \setminus U$.

$$h(\xi) = \frac{-1}{\left(1-\psi^{-1}(\xi)p\right)\psi^{-1}(\xi)\,\psi'\left(\psi^{-1}(\xi)\right)}$$

has two (simple) poles: one at $\psi\left(\frac{1}{p}\right)$ and one at $q_1 = 0$. (third factor never zero)

$$\operatorname{Res}(h; 0) = \frac{-1}{\left(1-\psi^{-1}(0)p\right)\psi'\left(\psi^{-1}(0)\right)} \frac{1}{\frac{1}{\psi'(\psi^{-1}(0))}} = -1$$

$$\operatorname{Res}\left(h; \psi(1/p)\right) = \frac{-1}{\psi^{-1}\left(\psi(1/p)\right)\psi'\left(\psi^{-1}(\psi(1/p))\right)} \frac{1}{\frac{-1}{\psi'(\psi^{-1}(\psi(1/p)))}\, p} = 1.$$

Suppose $p \notin \Gamma$.

$$n(\Gamma;p) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\varphi^{-1}\circ\psi^{-1}\circ\gamma_j} \frac{1}{z-p}\,dz = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\psi^{-1}\circ\gamma_j} \frac{-1}{(1-\zeta p)\zeta}\,d\zeta$$

$$= \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} \frac{-1}{\left(1-\psi^{-1}(\xi)p\right)\psi^{-1}(\xi)\,\psi'\left(\psi^{-1}(\xi)\right)}\,d\xi.$$

Suppose $p \in \mathbb{C} \setminus U$.

$$h(\xi) = \frac{-1}{\left(1-\psi^{-1}(\xi)p\right)\psi^{-1}(\xi)\,\psi'\left(\psi^{-1}(\xi)\right)}$$

has two (simple) poles: one at $\psi\left(\frac{1}{p}\right)$ and one at $q_1 = 0$. (third factor never zero)

$$\mathrm{Res}(h;0) = \frac{-1}{\left(1-\psi^{-1}(0)p\right)\psi'\left(\psi^{-1}(0)\right)}\,\frac{1}{\frac{1}{\psi'(\psi^{-1}(0))}} = -1$$

$$\mathrm{Res}\big(h;\psi(1/p)\big) = \frac{-1}{\psi^{-1}\left(\psi(1/p)\right)\psi'\left(\psi^{-1}(\psi(1/p))\right)}\,\frac{1}{\frac{-1}{\psi'(\psi^{-1}(\psi(1/p)))}\,p} = 1.$$

γ_1 goes around 0,

Suppose $p \notin \Gamma$.

$$n(\Gamma;p) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\varphi^{-1}\circ\psi^{-1}\circ\gamma_j} \frac{1}{z-p}\,dz = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\psi^{-1}\circ\gamma_j} \frac{-1}{(1-\zeta p)\zeta}\,d\zeta$$

$$= \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} \frac{-1}{\left(1-\psi^{-1}(\xi)p\right)\psi^{-1}(\xi)\,\psi'\left(\psi^{-1}(\xi)\right)}\,d\xi.$$

Suppose $p \in \mathbb{C} \setminus U$.

$$h(\xi) = \frac{-1}{\left(1-\psi^{-1}(\xi)p\right)\psi^{-1}(\xi)\,\psi'\left(\psi^{-1}(\xi)\right)}$$

has two (simple) poles: one at $\psi\left(\frac{1}{p}\right)$ and one at $q_1 = 0$. (third factor never zero)

$$\mathrm{Res}(h;0) = \frac{-1}{\left(1-\psi^{-1}(0)p\right)\psi'\left(\psi^{-1}(0)\right)} \frac{1}{\frac{1}{\psi'(\psi^{-1}(0))}} = -1$$

$$\mathrm{Res}\left(h;\psi(1/p)\right) = \frac{-1}{\psi^{-1}\left(\psi(1/p)\right)\psi'\left(\psi^{-1}(\psi(1/p))\right)} \frac{1}{\frac{-1}{\psi'(\psi^{-1}(\psi(1/p)))}\,p} = 1.$$

γ_1 goes around 0, some γ_j goes around $\psi\left(\frac{1}{p}\right) \in S$ (as $r < 1$ is large enough)

Suppose $p \notin \Gamma$.

$$n(\Gamma;p) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\varphi^{-1}\circ\psi^{-1}\circ\gamma_j} \frac{1}{z-p}\, dz = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\psi^{-1}\circ\gamma_j} \frac{-1}{(1-\zeta p)\zeta}\, d\zeta$$

$$= \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} \frac{-1}{\left(1-\psi^{-1}(\xi)p\right)\psi^{-1}(\xi)\,\psi'\left(\psi^{-1}(\xi)\right)}\, d\xi.$$

Suppose $p \in \mathbb{C} \setminus U$.

$$h(\xi) = \frac{-1}{\left(1-\psi^{-1}(\xi)p\right)\psi^{-1}(\xi)\,\psi'\left(\psi^{-1}(\xi)\right)}$$

has two (simple) poles: one at $\psi\left(\frac{1}{p}\right)$ and one at $q_1 = 0$. (third factor never zero)

$$\operatorname{Res}(h;0) = \frac{-1}{\left(1-\psi^{-1}(0)p\right)\psi'\left(\psi^{-1}(0)\right)} \frac{1}{\frac{1}{\psi'(\psi^{-1}(0))}} = -1$$

$$\operatorname{Res}\left(h;\psi(1/p)\right) = \frac{-1}{\psi^{-1}\left(\psi(1/p)\right)\psi'\left(\psi^{-1}(\psi(1/p))\right)} \frac{1}{\frac{-1}{\psi'(\psi^{-1}(\psi(1/p)))}\,p} = 1.$$

γ_1 goes around 0, some γ_j goes around $\psi\left(\frac{1}{p}\right) \in S$ (as $r < 1$ is large enough) $\quad\Rightarrow\quad n(\Gamma;p) = 0$

Suppose $p \in K$.

Suppose $p \in K$.

$$p \in K \quad \Rightarrow \quad \psi^{-1}(\xi) \neq \tfrac{1}{p} \text{ for all } \xi \in D,$$

Suppose $p \in K$.

$$p \in K \quad \Rightarrow \quad \psi^{-1}(\xi) \neq \tfrac{1}{p} \text{ for all } \xi \in D,$$

$$\Rightarrow \quad h(\xi) = \frac{-1}{\big(1 - \psi^{-1}(\xi)p\big)\,\psi^{-1}(\xi)\,\psi'\big(\psi^{-1}(\xi)\big)} \qquad \text{has only one pole } 0.$$

Suppose $p \in K$.

$p \in K \quad \Rightarrow \quad \psi^{-1}(\xi) \neq \frac{1}{p}$ for all $\xi \in D$,

$$\Rightarrow \quad h(\xi) = \frac{-1}{\left(1 - \psi^{-1}(\xi)p\right) \psi^{-1}(\xi)\, \psi'\left(\psi^{-1}(\xi)\right)} \qquad \text{has only one pole } 0.$$

$$n(\Gamma; p) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} h(\xi)\, d\xi$$

Suppose $p \in K$.

$p \in K \quad \Rightarrow \quad \psi^{-1}(\xi) \neq \frac{1}{p}$ for all $\xi \in D$,

$$\Rightarrow \quad h(\xi) = \frac{-1}{\left(1 - \psi^{-1}(\xi)p\right)\psi^{-1}(\xi)\,\psi'\left(\psi^{-1}(\xi)\right)} \quad \text{has only one pole } 0.$$

$$n(\Gamma; p) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} h(\xi)\,d\xi = \frac{1}{2\pi i} \int_{\gamma_1} h(\xi)\,d\xi$$

Suppose $p \in K$.

$p \in K \quad \Rightarrow \quad \psi^{-1}(\xi) \neq \frac{1}{p}$ for all $\xi \in D$,

$$\Rightarrow \quad h(\xi) = \frac{-1}{\left(1 - \psi^{-1}(\xi)p\right) \psi^{-1}(\xi)\, \psi'\left(\psi^{-1}(\xi)\right)} \quad \text{has only one pole } 0.$$

$$n(\Gamma; p) = \sum_{j=1}^{m} \frac{1}{2\pi i} \int_{\gamma_j} h(\xi)\, d\xi = \frac{1}{2\pi i} \int_{\gamma_1} h(\xi)\, d\xi = -\operatorname{Res}(h; 0) = 1. \quad \square$$

(γ_1 traverses the circle backwards)

Theorem

Let $U \subset \mathbb{C}$ *be a domain. Then* $\mathbb{C}_\infty \setminus U$ *is connected if and only if* U *is simply connected.*

Theorem

Let $U \subset \mathbb{C}$ be a domain. Then $\mathbb{C}_\infty \setminus U$ is connected if and only if U is simply connected.

Proof: Forward direction is done (we've just used it above).

Theorem

Let $U \subset \mathbb{C}$ be a domain. Then $\mathbb{C}_\infty \setminus U$ is connected if and only if U is simply connected.

Proof: Forward direction is done (we've just used it above).

Suppose $\mathbb{C}_\infty \setminus U$ is disconnected.

Theorem

Let $U \subset \mathbb{C}$ be a domain. Then $\mathbb{C}_\infty \setminus U$ is connected if and only if U is simply connected.

Proof: Forward direction is done (we've just used it above).

Suppose $\mathbb{C}_\infty \setminus U$ is disconnected.

Write $S \cup K = \mathbb{C}_\infty \setminus U$ where S and K are nonempty, closed, and disjoint.

Theorem

Let $U \subset \mathbb{C}$ be a domain. Then $\mathbb{C}_\infty \setminus U$ is connected if and only if U is simply connected.

Proof: Forward direction is done (we've just used it above).

Suppose $\mathbb{C}_\infty \setminus U$ is disconnected.

Write $S \cup K = \mathbb{C}_\infty \setminus U$ where S and K are nonempty, closed, and disjoint.

Assume $\infty \in S$.

Theorem

Let $U \subset \mathbb{C}$ be a domain. Then $\mathbb{C}_\infty \setminus U$ is connected if and only if U is simply connected.

Proof: Forward direction is done (we've just used it above).

Suppose $\mathbb{C}_\infty \setminus U$ is disconnected.

Write $S \cup K = \mathbb{C}_\infty \setminus U$ where S and K are nonempty, closed, and disjoint.

Assume $\infty \in S$.

$U' = U \cup K$ is open as S is closed,

Theorem

Let $U \subset \mathbb{C}$ be a domain. Then $\mathbb{C}_\infty \setminus U$ is connected if and only if U is simply connected.

Proof: Forward direction is done (we've just used it above).

Suppose $\mathbb{C}_\infty \setminus U$ is disconnected.

Write $S \cup K = \mathbb{C}_\infty \setminus U$ where S and K are nonempty, closed, and disjoint.

Assume $\infty \in S$.

$U' = U \cup K$ is open as S is closed, $\quad U' \subset \mathbb{C}$,

Theorem

Let $U \subset \mathbb{C}$ be a domain. Then $\mathbb{C}_\infty \setminus U$ is connected if and only if U is simply connected.

Proof: Forward direction is done (we've just used it above).

Suppose $\mathbb{C}_\infty \setminus U$ is disconnected.

Write $S \cup K = \mathbb{C}_\infty \setminus U$ where S and K are nonempty, closed, and disjoint.

Assume $\infty \in S$.

$U' = U \cup K$ is open as S is closed, $\quad U' \subset \mathbb{C}, \quad K \subset U'$ is compact.

Theorem

Let $U \subset \mathbb{C}$ be a domain. Then $\mathbb{C}_\infty \setminus U$ is connected if and only if U is simply connected.

Proof: Forward direction is done (we've just used it above).

Suppose $\mathbb{C}_\infty \setminus U$ is disconnected.

Write $S \cup K = \mathbb{C}_\infty \setminus U$ where S and K are nonempty, closed, and disjoint.

Assume $\infty \in S$.

$U' = U \cup K$ is open as S is closed, $\quad U' \subset \mathbb{C}, \quad K \subset U'$ is compact.

Apply lemma to find a cycle Γ in $U = U' \setminus K$ such that $n(\Gamma; z) = 1$ for all $z \in K$.

Theorem

Let $U \subset \mathbb{C}$ be a domain. Then $\mathbb{C}_\infty \setminus U$ is connected if and only if U is simply connected.

Proof: Forward direction is done (we've just used it above).

Suppose $\mathbb{C}_\infty \setminus U$ is disconnected.

Write $S \cup K = \mathbb{C}_\infty \setminus U$ where S and K are nonempty, closed, and disjoint.

Assume $\infty \in S$.

$U' = U \cup K$ is open as S is closed, $\quad U' \subset \mathbb{C}, \quad K \subset U'$ is compact.

Apply lemma to find a cycle Γ in $U = U' \setminus K$ such that $n(\Gamma; z) = 1$ for all $z \in K$.

In other words, Γ is not homologous to zero in U. $\square$

Exercise: Suppose $\{f_n\}$ is a sequence of holomorphic functions on an open set $U \subset \mathbb{C}$ that converges uniformly on compact subsets to a nonconstant $f : U \to \mathbb{C}$. Let $K \subset U$ be a compact set. Prove that for every open neighborhood V of K in U (so $K \subset V \subset U$) there exists a smaller open neighborhood W (so $K \subset W \subset V$) and an $N \in \mathbb{N}$ such that f and f_n have the same number of zeros in W for all $n \geq N$.