

Cultivating Complex Analysis: Inverses of holomorphic functions (5.6)

Jiří Lebl

Departemento pri Matematiko de Oklahoma Ŝtata Universitato

Let us restate the inverse function theorem.

Theorem (Inverse function theorem for holomorphic functions)

Suppose $U \subset \mathbb{C}$ is open, $f: U \rightarrow \mathbb{C}$ is holomorphic, $p \in U$, and $f'(p) \neq 0$. Then there exist open sets $V, W \subset \mathbb{C}$ such that $p \in V \subset U$, $f(V) = W$, the restriction $f|_V$ is injective (one-to-one), and hence a $g: W \rightarrow V$ exists such that $g(w) = (f|_V)^{-1}(w)$ for all $w \in W$. Furthermore, g is holomorphic and

$$g'(w) = \frac{1}{f'(g(w))} \quad \text{for all } w \in W.$$

Let us restate the inverse function theorem.

Theorem (Inverse function theorem for holomorphic functions)

Suppose $U \subset \mathbb{C}$ is open, $f: U \rightarrow \mathbb{C}$ is holomorphic, $p \in U$, and $f'(p) \neq 0$. Then there exist open sets $V, W \subset \mathbb{C}$ such that $p \in V \subset U$, $f(V) = W$, the restriction $f|_V$ is injective (one-to-one), and hence a $g: W \rightarrow V$ exists such that $g(w) = (f|_V)^{-1}(w)$ for all $w \in W$. Furthermore, g is holomorphic and

$$g'(w) = \frac{1}{f'(g(w))} \quad \text{for all } w \in W.$$

In other words, if f' is nonzero somewhere, f is injective near that point.

Let us restate the inverse function theorem.

Theorem (Inverse function theorem for holomorphic functions)

Suppose $U \subset \mathbb{C}$ is open, $f: U \rightarrow \mathbb{C}$ is holomorphic, $p \in U$, and $f'(p) \neq 0$. Then there exist open sets $V, W \subset \mathbb{C}$ such that $p \in V \subset U$, $f(V) = W$, the restriction $f|_V$ is injective (one-to-one), and hence a $g: W \rightarrow V$ exists such that $g(w) = (f|_V)^{-1}(w)$ for all $w \in W$. Furthermore, g is holomorphic and

$$g'(w) = \frac{1}{f'(g(w))} \quad \text{for all } w \in W.$$

In other words, if f' is nonzero somewhere, f is injective near that point.

Only local: $f(z) = z^2$ maps $\mathbb{C} \setminus \{0\}$ to itself, f' does not vanish, but f is 2-to-1 globally.

Real functions can be injective and the derivative can vanish:

$f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$, is injective but $f'(0) = 0$.

Real functions can be injective and the derivative can vanish:

$f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$, is injective but $f'(0) = 0$.

Holomorphic functions locally all behave like $z \mapsto z^k$, and that is injective only if $k = 1$.

Real functions can be injective and the derivative can vanish:

$f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$, is injective but $f'(0) = 0$.

Holomorphic functions locally all behave like $z \mapsto z^k$, and that is injective only if $k = 1$.

Lemma

If $U \subset \mathbb{C}$ is open and $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, then f' is never zero.

Real functions can be injective and the derivative can vanish:

$f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$, is injective but $f'(0) = 0$.

Holomorphic functions locally all behave like $z \mapsto z^k$, and that is injective only if $k = 1$.

Lemma

If $U \subset \mathbb{C}$ is open and $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, then f' is never zero.

Proof: Suppose f nonconstant and $f'(p) = 0$.

Real functions can be injective and the derivative can vanish:

$f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$, is injective but $f'(0) = 0$.

Holomorphic functions locally all behave like $z \mapsto z^k$, and that is injective only if $k = 1$.

Lemma

If $U \subset \mathbb{C}$ is open and $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, then f' is never zero.

Proof: Suppose f nonconstant and $f'(p) = 0$.

Let $\overline{\Delta_r(p)} \subset U$ be so that $f' \neq 0$ on $\Delta_r(p) \setminus \{p\}$, and $|f(z) - f(p)| > \delta > 0$ for $z \in \partial\Delta_r(p)$.

Real functions can be injective and the derivative can vanish:

$f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$, is injective but $f'(0) = 0$.

Holomorphic functions locally all behave like $z \mapsto z^k$, and that is injective only if $k = 1$.

Lemma

If $U \subset \mathbb{C}$ is open and $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, then f' is never zero.

Proof: Suppose f nonconstant and $f'(p) = 0$.

Let $\overline{\Delta_r(p)} \subset U$ be so that $f' \neq 0$ on $\Delta_r(p) \setminus \{p\}$, and $|f(z) - f(p)| > \delta > 0$ for $z \in \partial\Delta_r(p)$.

$z \mapsto f(z) - f(p)$ has a zero of multiplicity at least two.

Real functions can be injective and the derivative can vanish:

$f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$, is injective but $f'(0) = 0$.

Holomorphic functions locally all behave like $z \mapsto z^k$, and that is injective only if $k = 1$.

Lemma

If $U \subset \mathbb{C}$ is open and $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, then f' is never zero.

Proof: Suppose f nonconstant and $f'(p) = 0$.

Let $\overline{\Delta_r(p)} \subset U$ be so that $f' \neq 0$ on $\Delta_r(p) \setminus \{p\}$, and $|f(z) - f(p)| > \delta > 0$ for $z \in \partial\Delta_r(p)$.

$z \mapsto f(z) - f(p)$ has a zero of multiplicity at least two.

Let $w \in \Delta_\delta(f(p)) \setminus \{f(p)\}$

Real functions can be injective and the derivative can vanish:

$f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$, is injective but $f'(0) = 0$.

Holomorphic functions locally all behave like $z \mapsto z^k$, and that is injective only if $k = 1$.

Lemma

If $U \subset \mathbb{C}$ is open and $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, then f' is never zero.

Proof: Suppose f nonconstant and $f'(p) = 0$.

Let $\overline{\Delta_r(p)} \subset U$ be so that $f' \neq 0$ on $\Delta_r(p) \setminus \{p\}$, and $|f(z) - f(p)| > \delta > 0$ for $z \in \partial\Delta_r(p)$.

$z \mapsto f(z) - f(p)$ has a zero of multiplicity at least two.

Let $w \in \Delta_\delta(f(p)) \setminus \{f(p)\} \Rightarrow z \mapsto f(z) - w$ has at least two zeros (Rouché).

Real functions can be injective and the derivative can vanish:

$f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$, is injective but $f'(0) = 0$.

Holomorphic functions locally all behave like $z \mapsto z^k$, and that is injective only if $k = 1$.

Lemma

If $U \subset \mathbb{C}$ is open and $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, then f' is never zero.

Proof: Suppose f nonconstant and $f'(p) = 0$.

Let $\overline{\Delta_r(p)} \subset U$ be so that $f' \neq 0$ on $\Delta_r(p) \setminus \{p\}$, and $|f(z) - f(p)| > \delta > 0$ for $z \in \partial\Delta_r(p)$.

$z \mapsto f(z) - f(p)$ has a zero of multiplicity at least two.

Let $w \in \Delta_\delta(f(p)) \setminus \{f(p)\} \Rightarrow z \mapsto f(z) - w$ has at least two zeros (Rouché).

$f' \neq 0$ in $\Delta_r(p) \setminus \{p\}$

Real functions can be injective and the derivative can vanish:

$f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$, is injective but $f'(0) = 0$.

Holomorphic functions locally all behave like $z \mapsto z^k$, and that is injective only if $k = 1$.

Lemma

If $U \subset \mathbb{C}$ is open and $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, then f' is never zero.

Proof: Suppose f nonconstant and $f'(p) = 0$.

Let $\overline{\Delta_r(p)} \subset U$ be so that $f' \neq 0$ on $\Delta_r(p) \setminus \{p\}$, and $|f(z) - f(p)| > \delta > 0$ for $z \in \partial\Delta_r(p)$.

$z \mapsto f(z) - f(p)$ has a zero of multiplicity at least two.

Let $w \in \Delta_\delta(f(p)) \setminus \{f(p)\} \Rightarrow z \mapsto f(z) - w$ has at least two zeros (Rouché).

$f' \neq 0$ in $\Delta_r(p) \setminus \{p\} \Rightarrow$ zeros of $z \mapsto f(z) - w$ are simple

Real functions can be injective and the derivative can vanish:

$f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$, is injective but $f'(0) = 0$.

Holomorphic functions locally all behave like $z \mapsto z^k$, and that is injective only if $k = 1$.

Lemma

If $U \subset \mathbb{C}$ is open and $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, then f' is never zero.

Proof: Suppose f nonconstant and $f'(p) = 0$.

Let $\overline{\Delta_r(p)} \subset U$ be so that $f' \neq 0$ on $\Delta_r(p) \setminus \{p\}$, and $|f(z) - f(p)| > \delta > 0$ for $z \in \partial\Delta_r(p)$.

$z \mapsto f(z) - f(p)$ has a zero of multiplicity at least two.

Let $w \in \Delta_\delta(f(p)) \setminus \{f(p)\} \Rightarrow z \mapsto f(z) - w$ has at least two zeros (Rouché).

$f' \neq 0$ in $\Delta_r(p) \setminus \{p\} \Rightarrow$ zeros of $z \mapsto f(z) - w$ are simple

$\Rightarrow f(z) - w$ has at least two distinct zeros

Real functions can be injective and the derivative can vanish:

$f: \mathbb{R} \rightarrow \mathbb{R}$, $f(x) = x^3$, is injective but $f'(0) = 0$.

Holomorphic functions locally all behave like $z \mapsto z^k$, and that is injective only if $k = 1$.

Lemma

If $U \subset \mathbb{C}$ is open and $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, then f' is never zero.

Proof: Suppose f nonconstant and $f'(p) = 0$.

Let $\overline{\Delta_r(p)} \subset U$ be so that $f' \neq 0$ on $\Delta_r(p) \setminus \{p\}$, and $|f(z) - f(p)| > \delta > 0$ for $z \in \partial\Delta_r(p)$.

$z \mapsto f(z) - f(p)$ has a zero of multiplicity at least two.

Let $w \in \Delta_\delta(f(p)) \setminus \{f(p)\} \Rightarrow z \mapsto f(z) - w$ has at least two zeros (Rouché).

$f' \neq 0$ in $\Delta_r(p) \setminus \{p\} \Rightarrow$ zeros of $z \mapsto f(z) - w$ are simple

$\Rightarrow f(z) - w$ has at least two distinct zeros $\Rightarrow f$ is not injective.



We can actually compute the inverse:

Lemma

If $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, and $\overline{\Delta_r(p)} \subset U$. Then for all $w \in f(\Delta_r(p))$,

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{\partial \Delta_r(p)} \frac{f'(z)z}{f(z) - w} dz.$$

We can actually compute the inverse:

Lemma

If $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, and $\overline{\Delta_r(p)} \subset U$. Then for all $w \in f(\Delta_r(p))$,

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{\partial \Delta_r(p)} \frac{f'(z)z}{f(z) - w} dz.$$

Proof: Fix $w \in f(\Delta_r(p))$ and $\zeta \in \Delta_r(p)$ such that $f(\zeta) = w$.

We can actually compute the inverse:

Lemma

If $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, and $\overline{\Delta_r(p)} \subset U$. Then for all $w \in f(\Delta_r(p))$,

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{\partial \Delta_r(p)} \frac{f'(z)z}{f(z) - w} dz.$$

Proof: Fix $w \in f(\Delta_r(p))$ and $\zeta \in \Delta_r(p)$ such that $f(\zeta) = w$.

f' is never zero, so $z \mapsto f(z) - w$ has a simple zero at $z = \zeta$.

We can actually compute the inverse:

Lemma

If $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, and $\overline{\Delta_r(p)} \subset U$. Then for all $w \in f(\Delta_r(p))$,

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{\partial \Delta_r(p)} \frac{f'(z)z}{f(z) - w} dz.$$

Proof: Fix $w \in f(\Delta_r(p))$ and $\zeta \in \Delta_r(p)$ such that $f(\zeta) = w$.

f' is never zero, so $z \mapsto f(z) - w$ has a simple zero at $z = \zeta$.

By the residue theorem

$$\frac{1}{2\pi i} \int_{\partial \Delta_r(p)} \frac{f'(z)z}{f(z) - w} dz = \operatorname{Res} \left(\frac{f'(z)z}{f(z) - w}; \zeta \right)$$

We can actually compute the inverse:

Lemma

If $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, and $\overline{\Delta_r(p)} \subset U$. Then for all $w \in f(\Delta_r(p))$,

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{\partial \Delta_r(p)} \frac{f'(z)z}{f(z) - w} dz.$$

Proof: Fix $w \in f(\Delta_r(p))$ and $\zeta \in \Delta_r(p)$ such that $f(\zeta) = w$.

f' is never zero, so $z \mapsto f(z) - w$ has a simple zero at $z = \zeta$.

By the residue theorem

$$\frac{1}{2\pi i} \int_{\partial \Delta_r(p)} \frac{f'(z)z}{f(z) - w} dz = \operatorname{Res} \left(\frac{f'(z)z}{f(z) - w}; \zeta \right) = \frac{f'(\zeta)\zeta}{f'(\zeta)} = \zeta = f^{-1}(w). \quad \square$$

We can actually compute the inverse:

Lemma

If $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, and $\overline{\Delta_r(p)} \subset U$. Then for all $w \in f(\Delta_r(p))$,

$$f^{-1}(w) = \frac{1}{2\pi i} \int_{\partial \Delta_r(p)} \frac{f'(z)z}{f(z) - w} dz.$$

Proof: Fix $w \in f(\Delta_r(p))$ and $\zeta \in \Delta_r(p)$ such that $f(\zeta) = w$.

f' is never zero, so $z \mapsto f(z) - w$ has a simple zero at $z = \zeta$.

By the residue theorem

$$\frac{1}{2\pi i} \int_{\partial \Delta_r(p)} \frac{f'(z)z}{f(z) - w} dz = \operatorname{Res} \left(\frac{f'(z)z}{f(z) - w}; \zeta \right) = \frac{f'(\zeta)\zeta}{f'(\zeta)} = \zeta = f^{-1}(w). \quad \square$$

Consequently, f^{-1} is holomorphic without even using the inverse function theorem.

Theorem

If $U \subset \mathbb{C}$ is open and $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, then $f(U)$ is open, f' is never zero on U , and $f^{-1}: f(U) \rightarrow U$ is holomorphic.

Theorem

If $U \subset \mathbb{C}$ is open and $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, then $f(U)$ is open, f' is never zero on U , and $f^{-1}: f(U) \rightarrow U$ is holomorphic.

Proof: $f(U)$ is open by the open mapping theorem.

Theorem

If $U \subset \mathbb{C}$ is open and $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, then $f(U)$ is open, f' is never zero on U , and $f^{-1}: f(U) \rightarrow U$ is holomorphic.

Proof: $f(U)$ is open by the open mapping theorem.

By one of the lemmas, f' is never zero on U .

Theorem

If $U \subset \mathbb{C}$ is open and $f: U \rightarrow \mathbb{C}$ is holomorphic and injective, then $f(U)$ is open, f' is never zero on U , and $f^{-1}: f(U) \rightarrow U$ is holomorphic.

Proof: $f(U)$ is open by the open mapping theorem.

By one of the lemmas, f' is never zero on U .

By the other (or IFT), f^{-1} is holomorphic.

