Cultivating Complex Analysis: Matrix representation of complex numbers (1.1.4)

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Easy exercise: If M is $z \mapsto \xi z$, then $\det(M) = |\xi|^2$. If M is $z \mapsto \xi z + \zeta \bar{z}$, then $\det(M) = |\xi|^2 - |\zeta|^2$.

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The derivative is a matrix of partial derivatives, so to be of the form $\left[\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right]$ is a system of partial differential equations (PDE): the Cauchy–Riemann equations.