Cultivating Complex Analysis: Laurent series (4.4 part 2)

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Theorem (Existence of Laurent series)

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converging uniformly absolutely on compact subsets of $\operatorname{ann}(p; r_1, r_2)$. *The numbers* c_n *are given by*

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z-p)^{n+1}}\, dz,$$

where γ *is any circle of radius* s, $r_1 < s < r_2$, *centered at* p *oriented counterclockwise.*

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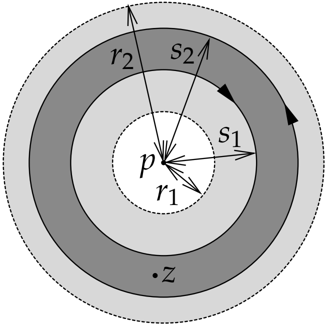

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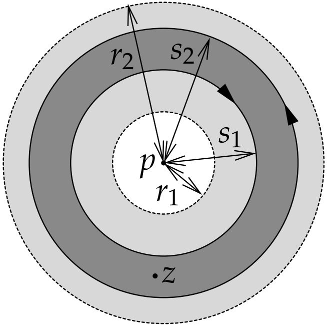

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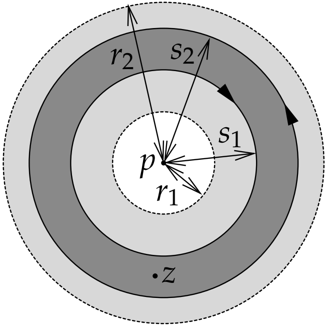

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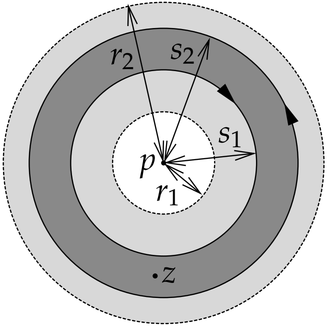

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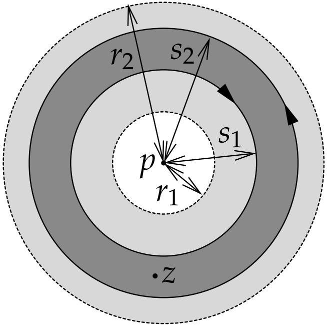

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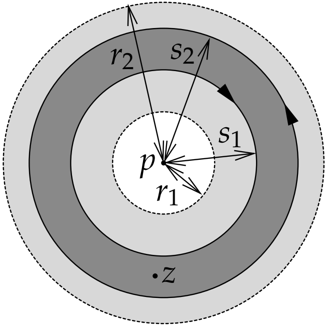

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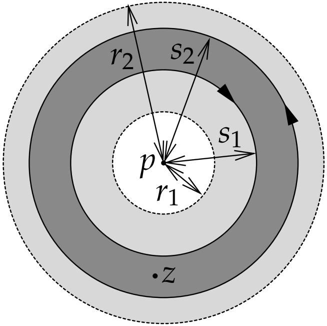

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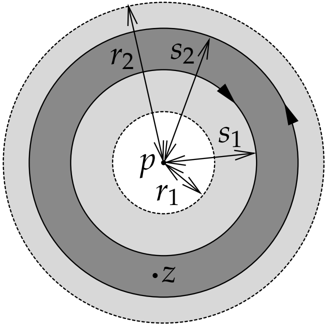

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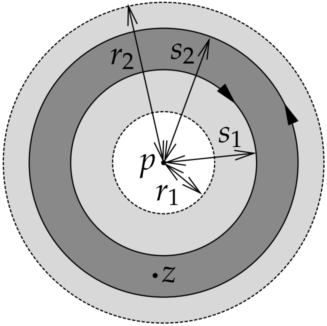

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We will expand the two integrals separately.

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We can swap the series limit with the integral as the convergence is uniform on the circle.

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For any $\epsilon > 0$, the geometric series used for the first part converges uniformly absolutely when $\left|\frac{z-p}{\zeta-p}\right| = \frac{|z-p|}{s_2} \leq 1 - \epsilon$.

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$$= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} d_n \int_{\partial \Delta_s(p)} (\zeta - p)^{n-m-1} \, d\zeta = d_m.$$

(The last equality because $\int_{\partial \Delta_s(p)} (\zeta - p)^{n-m-1} \, d\zeta \neq 0$ only when $n = m$.) $\square$

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Proof: Exercise.

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Exercise: Expand the function $f(z) = \frac{1}{(z-1)(z-2)}$ in the sets $\operatorname{ann}(0; 0, 1)$, $\operatorname{ann}(0; 1, 2)$, and $\operatorname{ann}(0; 2, \infty)$.