# Cultivating Complex Analysis: Cauchy for star-like sets (3.2.3)

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## Definition

A set $U \subset \mathbb{C}$ is called *star-like* (or *star-like with respect to* $z_0$) if there exists a point $z_0 \in U$ such that the segment $[z_0, z] \subset U$ for every $z \in U$.

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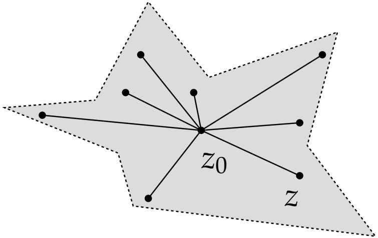

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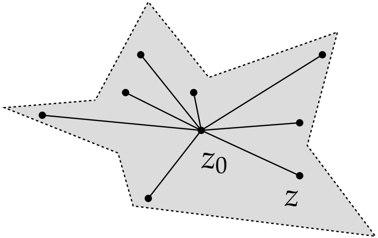


A convex set is star-like, but not vice versa.

## Proposition

*Suppose* $U \subset \mathbb{C}$ *is open and star-like,* $f : U \longrightarrow \mathbb{C}$ *is continuous, and*

$$\int_{\partial T} f(z)\, dz = 0 \qquad \textit{for every triangle } T \subset U.$$

*Then* $f$ *has a primitive: There exists a holomorphic* $F : U \longrightarrow \mathbb{C}$ *such that* $F' = f$.

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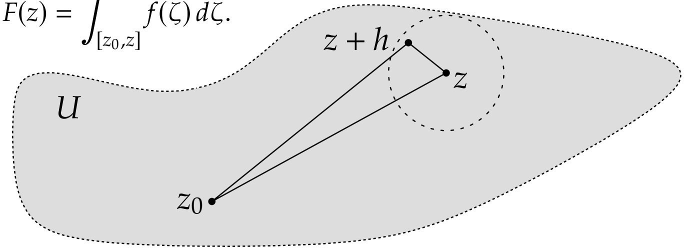


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$U$ is star-like w.r.t. $z_0 \Rightarrow$ the entire triangle with vertices $z_0$, $z$, and $z + h$ is in $U$.

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By continuity of $f$ at $z$,

$$\lim_{h\to 0}\frac{F(z+h)-F(z)}{h} = f(z). \quad \square$$

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*Suppose $U \subset \mathbb{C}$ is open and star-like, $f : U \to \mathbb{C}$ is holomorphic, and $\Gamma$ is a cycle in $U$. Then*

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**Remark:** $\mathbb{C}$-valued function gives a vector-field on $\mathbb{R}^2$.
The corollary is a special case of a theorem from vector calculus:

*In a star-like domain $U \subset \mathbb{R}^2$, if a $C^1$ vector field $(u, v) : U \to \mathbb{R}^2$ satisfies $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ (irrotational), then there exists a real-valued $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla f = (u, v)$ (conservative vector field).*