Cultivating Complex Analysis: Automorphisms of the disc (3.5.2)

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For every $a \in \mathbb{D}$,

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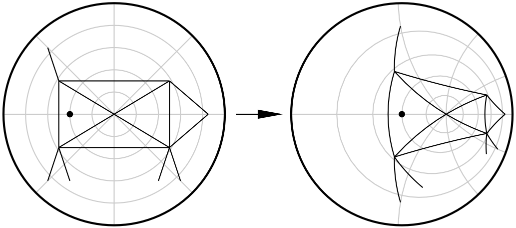
Proof is an exercise.

Here is what an automorphism φ_a does ($a = -0.4$).



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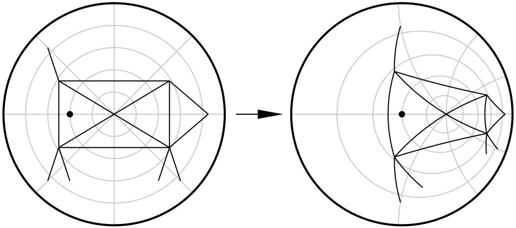

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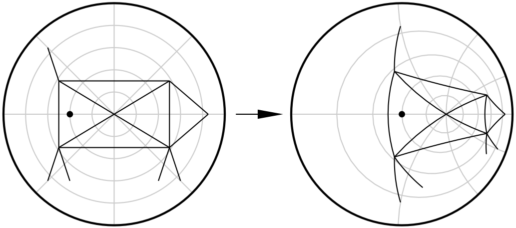

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If $f \in \mathrm{Aut}(\mathbb{D})$, then there exists an $a \in \mathbb{D}$ and $\theta \in \mathbb{R}$ such that

$$f(z) = e^{i\theta} \frac{z - a}{1 - \bar{a}z} = e^{i\theta} \varphi_a(z).$$

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Apply φ_{-a} to both sides of $e^{i\theta} z = \varphi_a \circ f$ to find $f(z) = \varphi_{-a}(ze^{i\theta}) = e^{i\theta}\varphi_{-ae^{-i\theta}}(z)$. □

Some good exercises to try:

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Hint: The idea is to show that you can divide by finitely many $\varphi_a(z)$ for various a until you get something that has no zeros in $\mathbb{D}$ and will have to be a constant.

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Equality $\Rightarrow f(z) = \varphi_{-b}\big(e^{i\theta}\varphi_a(z)\big)$ for some $\theta \in \mathbb{R}$.