Cultivating Complex Analysis: Power series (2.3 part 1)

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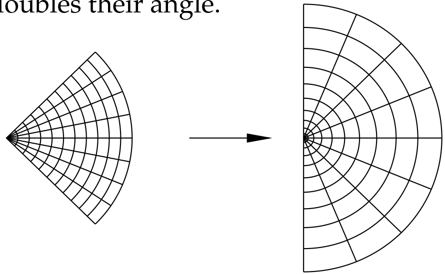
to the closed right half-plane
$\{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$.



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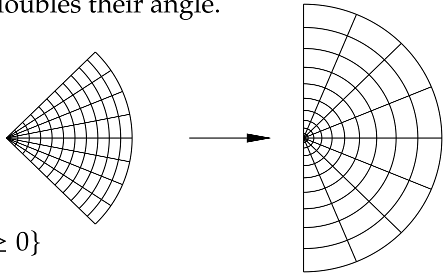

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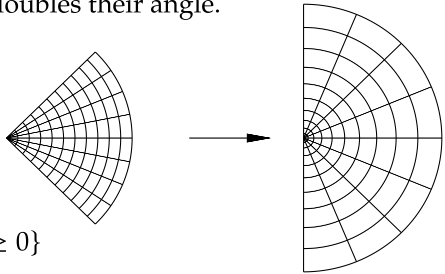

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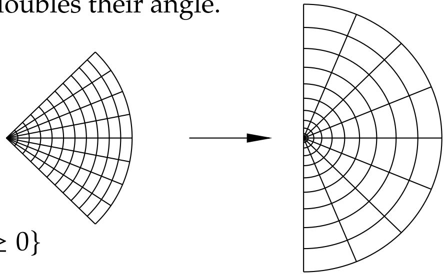

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For each $w = re^{i\theta} \neq 0$, there are n distinct n^{th} roots

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They are equally spaced out on a circle of radius $r^{1/n}$.

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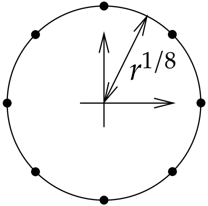

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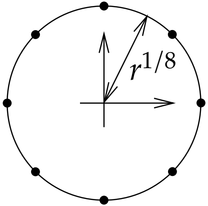

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The roots of $w = 1$ are called the *roots of unity*.

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If $z \neq 1$ is such that $|z| = 1$, then z^n diverges as $n \longrightarrow \infty$.

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We say a power series is *convergent* if it converges for any $z \neq p$.

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which follows by expanding $(1-z)(1+z+z^2+\cdots+z^m)$. □

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Thus, an absolutely convergent series converges.